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# Forecasting the Performance of the Energy Sector at the Saudi Stock Exchange Market by Using GBM and GFBM Models

**Mohammed Alhagyan**

Mathematics Department, College of Humanities and Science, Prince Sattam bin Abdulaziz University, Al-Kharj 11912, Saudi Arabia; m.alhagyan@psau.edu.sa

**Abstract:** Future index prices are viewed as a critical issue for any trader and investor. In the literature, various models have been developed for forecasting index prices. For example, the geometric Brownian motion (GBM) model is one of the most popular tools. This work examined four types of GBM models in terms of the presence of memory and the kind of volatility estimations. These models include the classical GBM model with memoryless and constant volatility assumptions, the SVGBM model with memoryless and stochastic volatility assumptions, the GFBM model with memory and constant volatility assumptions, and the SVGFBM model with memory and stochastic volatility assumptions. In this study, these models were utilized in an empirical study to forecast the future index price of the energy sector in the Saudi Stock Exchange Market. The assessment was led by utilizing two error standards, the mean square error (MSE) and mean absolute percentage error (MAPE). The results show that the SVGFBM model demonstrates the highest accuracy, resulting in the lowest MSE and MAPE, while the GBM model was the least accurate of all the models under study. These results affirm the benefits of combining memory and stochastic volatility assumptions into the GBM model, which is also supported by the findings of numerous earlier studies. Furthermore, the findings of this study show that GFBM models are more accurate than GBM models, regardless of the type of volatility. Furthermore, under the same type of memory, the models with a stochastic volatility assumption are more accurate than the corresponding models with a constant volatility assumption. In general, all models considered in this work showed a high accuracy, with MAPE $\leq 10\%$. This indicates that these models can be applied in real financial environments. Based on the results of this empirical study, the future of the energy sector in Saudi Arabia is forecast to be predictable and stable, and we urge financial investors and stockholders to trade and invest in this sector.

**Keywords:** energy sector; GBM; GFBM; stochastic volatility



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## 1. Introduction

The Kingdom of Saudi Arabia is the largest producer and exporter of oil in the world. For this reason, Saudi Arabia recognizes the significance of diversifying its energy mix to maintain long-term economic prosperity. Both domestic and foreign partners in the energy sector play an important role in the Kingdom's transition toward a sustainable and renewable future. Therefore, the energy sector is very important in the Saudi Exchange stock market. The energy sector in Saudi Arabia consists of six companies with a total capital exceeding SAR 7 trillion (USD 1.86 trillion). Hence, there are a large number of investors, traders, and speculators involved in this sector. Therefore, the future performance of this sector is considered a fundamental issue for all types of traders to gain profits and avoid possible losses. For this purpose, a need for a tool that can forecast future prices as precisely as possible arose. There are two approaches to describing and forecasting index prices: discrete time setting and continuous time setting. Since the intuitive setting for market trading is typically continuous, we are motivated to focus on the study of continuous time setting in a financial environment. Scholars have proposed several continuous models as tools that employ historical data to forecast future prices, such as random walk, jump-diffusion, Brownian motion process, and geometric Brownian motion (GBM) models. This work investigates some of the GBM models by incorporating the assumptions of stochastic volatility and memory.

## 2. Geometric Brownian Motion Models

The econophysics concept of the GBM model explains the nature of stock price randomness and arbitrary fluctuation calculations more accurately (Kumar et al. 2024). Occasionally, the GBM model has been called "the standard model of finance" (Ibe 2013), where it is employed in forecasting the price of a stock over time. The GBM model is the adapted version of the Brownian motion (BM) process.

**Definition 1.** *A Wiener Process or Brownian motion (BM) is a stochastic process $B_t$ that satisfies the following conditions*:

*i.* $B_t$ *is a continuous function of time with* $B_0 = 0$.
*ii.* $B_t$ *has independent increments* ($B_t - B_s$ *and* $B_v - B_u$ *are independent for all* $v > v > t > s$).
*iii.* $B_t - B_s \sim N(0,\ t-s)$ *for all* $t > s$.

According to the above definition, the BM process is continuous everywhere, but it is not differentiable anywhere. The BM is self-similar (i.e., if any part of the BM time-series trajectory is like the entire trajectory). If the BM touches any specific value, it will return to this value again an infinite number of times.

These properties encouraged Ross (1999) to model stock prices directly depending on the BM. However, this way of modeling has faced reasonable blame because of the nature of the BM, which permits the price to be negative when the stock price is supposed to be a normal random variable. As a treatment of this issue, the GBM model has been presented as an adaptation of the BM.

**Model 1. Geometric Brownian Motion (GBM)**

The stochastic process $X_t$ is said to follow GBM if it satisfies the following stochastic differential equation (SDE):

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{1}$$

where $B_t$ is a BM, and $\mu$ and $\sigma$ are drift and volatility, respectively. The general solution of this SDE is provided by the following equation:

$$X_t = X_0 exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t \right\} \tag{2}$$

where $X_0$ is an initial value.

The GBM model is a non-negative variation of the BM. Consequently, the GBM model can be employed in many financial applications, such as index prices, exchange rates, option pricing, mortgage insurance, and value at risk. The GBM model is valuable for modeling random price fluctuation over time and investigating a commodity's price performance. Hence, the GBM model is used widely to predict future prices.

The classical GBM model assumes the independence of prices. Meanwhile, many researchers have drawn attention to the existence of memory in the GBM model, for example Han et al. (2020), Rejichi and Aloui (2012), Alhagyan and Yassen (2023), Painter (1998), Alhagyan (2018), Grau-Carles (2000), and Kim et al. (2020).

The results of these studies indicated the necessity of further developing the GBM model by incorporating the properties of memory. The improved model is called the geometric fractional Brownian motion (GFBM) model. The GFBM model is obtained by replacing the classical BM process (no memory) with a developed process called the fractional Brownian motion (FBM) process (with memory).

**Definition 2.** *The fractional Brownian motion (FBM), $\{B_H(t)\}$, with Hurst parameter $H \in (0,1)$ is a centered Gaussian process whose paths are continuous with probability 1, and its distribution is defined by the following covariance structure:*

$$E[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

The correlation between the increments of FBM ($B_H(t) - B_H(s)$ and $B_H(v) - B_H(u)$ for all $v > v > t > s$) fluctuates conveniently with self-similarity or Hurst parameter H. The Hurst parameter name refers to Harold Edwin Hurst (1880–1978), who examined the erratic rainfall and drought circumstances along the Nile River over a long period. Three types of memories appeared according to the value of $H$: a short memory when $0 < H < 0.5$, no memory if $H = 0.5$, and a long memory when $0.5 < H < 1$. Now, by replacing BM in Equation (1) with FBM, we obtained the GFBM model that is presented as follows:

**Model 2. Geometric Fractional Brownian Motion (GFBM)**

The stochastic process $X_t$ is said to follow GFBM if it satisfies the following SDE:

$$dX_t = \mu X_t dt + \sigma X_t\, dB_{H_1}(t) \tag{3}$$

where $B_{H_1}(t)$ is FBM, and $\mu$ and $\sigma$ are drift and volatility, respectively. The solution is provided by the following equation:

$$X(t) = X_0 exp\left[\left(\mu - \frac{1}{2}\sigma^2 t^{2H_1-1}\right)t + \sigma B_{H_1}(t)\right] \tag{4}$$

where $X_0$ is an arbitrary initial value.

The GFBM model is considered a developed version of the GBM model, so it can be employed in the same financial applications, such as option pricing (Misiran 2010; Misiran et al. 2012; Alhagyan et al. 2016), index prices (Alhagyan and Alduais 2020; Abbas and Alhagyan 2022; Xiao et al. 2015), value at risk (Alhagyan et al. 2021; Wang et al. 2017), exchange rate (Gözgör 2013; Mansour and Ayasrah 2022; Alhagyan 2022), and mortgage insurance (Bardhan et al. 2006; Alhagyan et al. 2021; Chen et al. 2013).

A constant assumption of volatility ($\sigma$) was used in the GBM models to simplify calculations. However, this assumption was not supported by some empirical studies, such as those of Stein (1989), Bakshi et al. (2000), and Aït-Sahalia and Lo (1998), which concluded that the assumption of constant volatility does not describe the real-life situation accurately. For this reason, there have been many attempts to address this issue by replacing the constant volatility ($\sigma$) in the deterministic function of the stochastic process or volatility process $\sigma(Y_t)$ in GBM models, where $Y_t$ is the solution of other stochastic differential equations (SDEs) that is driven by other stochastic volatility noise. Examples of this research are reported in the efforts of Scott (1987), Hull and White (1987), Alhagyan et al. (2016), Stein and Stein (1991), Heston (1993), Alhagyan and Yassen (2023), Hagan et al. (2002), Alhagyan (2022), Comte and Renault (1998), Chronopoulou and Viens (2012a, 2012b), Wang and Zhang (2014), and Alhagyan (2018).

SV models are considerable in the environment of the financial market because of their ability to capture the effect of time-varying volatility. SV models permit both volatility and the common dependence between variables to fluctuate over time. This implies that SV models have two sources of randomness. Table 1 presents some SDE equations describing the stochastic process $Y_t$.

**Table 1.** Models of stochastic processes describing $Y_t$ in SV models.

| Name | Model |
|---|---|
| Lognormal process | $dY_t = \alpha Y_t dt + \beta Y_t dB_{2t}$ |
| Cox–Ingersoll–Ross (CIR) process | $dY_t = \theta(\omega - Y_t)dt + \xi\sqrt{Y_t}dB_{2t}$ |
| Ornstein–Uhlenbeck (OU) process | $dY_t = \alpha(m - Y_t)dt + \beta dB_{2t}$ |
| Not mean reverting process | $dY_t = \alpha Y_t dB_{2t}$ |
| Fractional Ornstein–Uhlenbeck (FOU) process | $dY_t = \alpha(m - Y_t)dt + \beta dB_{H_2}(t)$ |

In what follows, two models of SV under study are presented.

**Model 3. GBM perturbed using FOU (SVGBM)**

The stochastic process $X_t$ is said to follow GBM perturbed by SV (FOU) if it satisfies the following SDE:

$$dX_t = \mu X_t dt + \sigma(Y_t)X_t dB_{1t} \tag{5}$$

where $\mu$ is a mean of return, $Y_t$ is a stochastic process, $B_{1t}$ is a BM, and $\sigma(Y_t) = Y_t$ is a deterministic function. Let the dynamics of volatility $Y_t$ follow the fractional Ornstein–Uhlenbeck (FOU) process, which is the solution of the following SDE:

$$dY_t = \alpha(m - Y_t)dt + \beta dB_{H_2}(t) \tag{6}$$

where $\alpha$, $\beta$, and $m$ are the mean reverting of volatility, volatility of volatility, and mean of volatility, respectively. $B_{H_2}(t)$ is an FBM which is independent from $B_{1t}$.

**Model 4. GFBM perturbed using FOU (SVGFBM)**

The stochastic process $X_t$ is said to follow a GFBM perturbed by SV (FOU) if it satisfies the following SDE:

$$dX_t = \mu\, X_t dt + \sigma(Y_t)X_t dB_{H_1}(t) \tag{7}$$

where $\mu$ is the mean of return, $Y_t$ is a stochastic process, $B_{H_1}(t)$ is an FBM with Hurst index $H_1$, *and* $\sigma(Y_t) = Y_t$ is a deterministic function. Let the dynamics of volatility $Y_t$ be described by the fractional Ornstein–Uhlenbeck (FOU) p rocess, which is the solution of the following SDE:

$$dY_t = \alpha(m - Y_t)dt + \beta dB_{H_2}(t) \tag{8}$$

where $\alpha$, $\beta$, and $m$ are constant parameters that represent the mean reverting of volatility, volatility of volatility, and mean of volatility, respectively. $B_{H_2}(t)$ is an FBM, which is independent from $B_{H_1}(t)$.

## 3. Forecasting

This study employs the four models mentioned in the previous section to forecast the future index prices of seven companies in the energy sector in Saudi Arabia depending on historical data. In this empirical study, we examine the influence of incorporating both memory and stochastic volatility into the GBM model.

We relied on two measures of error to evaluate and compare the performance of each model under study. These measures are the mean square error ($MSE$) and mean absolute percentage error ($MAPE$):

$$MSE = \frac{\sum_{i=1}^{n}(Y_i - F_i)^2}{n} \text{ and } MAPE = \frac{\sum_{i=1}^{n}\frac{|Y_i - F_i|}{Y_i}}{n},$$

where $F_i$ and $Y_i$ represent the forecast and actual price at day $i$, respectively, while $n$ represents the total forecasting days. Lawrence et al. (2009) recapped the judgment on any forecasting method using MAPE in Table 2.

**Table 2.** MAPE judgment accuracy of forecasting method.

| Judgment of Accuracy | MAPE |
|---|---|
| Highly accurate | $MAPE < 10\%$ |
| Good accuracy | $10\% \leq MAPE < 20\%$ |
| Reasonable | $20\% \leq MAPE < 50\%$ |
| Inaccurate | $MAPE \geq 50\%$ |

*3.1. Description of Historical Data*

The historical energy index data are available online at https://www.saudiexchange.sa (accessed on 12 February 2024). The total working days are 43 days, from 1 November 2023 to 31 December 2023. To avoid high fluctuation in data, the return series is considered in logarithm (i.e., $r_n = \ln(S/S_{n-1})$). Figures 1 and 2 show the close prices and their return series.

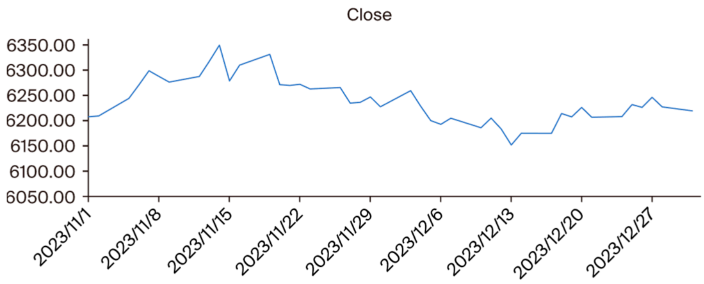


**Figure 1.** Daily energy index close price.

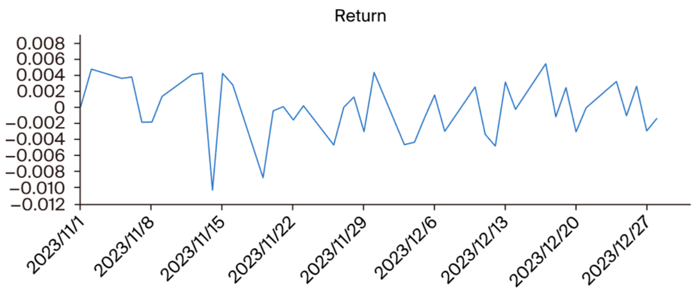


**Figure 2.** Daily returns of energy index.

*3.2. Forecasting and Evaluation*

According to the historical data of energy sector indices, all parameters involved in the four models under study were calculated by using Mathematica 10 software and then employed to compute constant volatility and stochastic volatility. Table 3 lists all computed parameters and volatilities.

**Table 3.** Parameter and volatility values.

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $H_1$ | 0.3279 | $\beta$ | 0.00002 |
| $H_2$ | 0.2520 | $m$ | 0.00001 |
| $\mu$ | 0.00004 | $\alpha$ | 4.62705 |
| **Computed Volatility** | | | |
| $\sigma$ | 0.0036 | | |
| $\sigma(Y_t)$ | 0.0010 | | |

Then, the close price of the next month (January 2024) was forecast based on the values of the parameters in Table 3. The forecasting was computed using the following four models: GBM, GFBM, SVGBM, and SVGFBM. Table 4 shows the accuracy of each model. Meanwhile, Table 5 shows the forecast prices beside the actual prices of the energy indices.

**Table 4.** The accuracy ranking level of the forecasting model is based on MAPE and MSE values.

| Model | MAPE | MSE |
|---|---|---|
| SVGFBM | 2.753% | 44,861 |
| GFBM | 2.758% | 44,921 |
| SVGBM | 2.759% | 44,946 |
| GBM | 2.887% | 48,457 |

**Table 5.** Actual and forecast prices.

| Date | Actual | GBM | GFBM | SV GBM | SV GFBM |
|---|---|---|---|---|---|
| 1/1/2024 | 6231.24 | 6240.65 | 6230.96 | 6228.20 | 6231.33 |
| 1/2/2024 | 6234.45 | 6243.74 | 6231.77 | 6231.27 | 6231.74 |
| 1/3/2024 | 6208.98 | 6244.15 | 6230.47 | 6231.68 | 6231.56 |
| 1/4/2024 | 6231.12 | 6244.03 | 6228.52 | 6231.59 | 6231.21 |
| 1/7/2024 | 6264.88 | 6243 | 6232.38 | 6230.56 | 6232.46 |
| 1/8/2024 | 6298.10 | 6245.29 | 6232.93 | 6232.84 | 6232.8 |
| 1/9/2024 | 6258.55 | 6247.55 | 6230.89 | 6235.09 | 6232.42 |
| 1/10/2024 | 6194.68 | 6243.82 | 6233.67 | 6235.34 | 6233.37 |
| 1/11/2024 | 6184.72 | 6243.25 | 6233.11 | 6230.77 | 6233.4 |
| 1/14/2024 | 6185.47 | 6244.28 | 6233.75 | 6234.82 | 6233.76 |
| 1/15/2024 | 6104.82 | 6245.72 | 6235.91 | 6233.24 | 6234.54 |
| 1/16/2024 | 6026.77 | 6246.9 | 6233.81 | 6234.45 | 6234.14 |
| 1/17/2024 | 5989.73 | 6246.56 | 6235.95 | 6234.11 | 6234.92 |
| 1/18/2024 | 6014.71 | 6246.06 | 6233.42 | 6236.59 | 6234.4 |
| 1/21/2024 | 6030.20 | 6246.09 | 6235.19 | 6236.62 | 6235.07 |
| 1/22/2024 | 5973.57 | 6248.83 | 6235.01 | 6236.35 | 6233.2 |
| 1/23/2024 | 5978.97 | 6247.24 | 6233.49 | 6236.75 | 6234.97 |
| 1/24/2024 | 5987.34 | 6246.39 | 6236.14 | 6236.91 | 6235.89 |
| 1/25/2024 | 5958.15 | 6246.5 | 6236.04 | 6234.04 | 6233.04 |
| 1/28/2024 | 5932.13 | 6247.66 | 6234.01 | 6235.19 | 6235.66 |
| 1/29/2024 | 5904.70 | 6243.1 | 6237.73 | 6236.62 | 6236.87 |
| 1/30/2024 | 5896.26 | 6247.03 | 6235.71 | 6234.57 | 6235.5 |
| 1/31/2024 | 5766.84 | 6247.19 | 6236.92 | 6234.71 | 6237.01 |

In light of the smaller MSE and MAPE values, the results show that the SVGFBM model has the highest accuracy. This was due to the presence of two sources of memory ($H_1$ and $H_2$) along with the stochastic volatility assumption. The GBM model placed last because of the existence of one source of randomness in addition to the constant volatility assumption. The outcomes demonstrated that the models under memory assumption (SVGFBM and GFBM models) are more appropriate for forecasting future stock costs than the models without memory (SVGBM and GBM models).

As per Lawrence's table (Table 2), all models achieved MAPE < 10%, which indicates a high forecasting accuracy in these models. Moreover, one can observe that the MSE values of the SVGFBM, GFBM, and SVGBM models are close together in number, while the MSE value of the GBM model is larger. These outcomes are in line with many experimental studies, for instance, Willinger et al. (1999), Rejichi and Aloui (2012), Alhagyan (2022), Painter (1998), Alhagyan and Yassen (2023), and Abbas and Alhagyan (2022).

Figure 3 illustrates the comparison between the actual close prices versus forecast close prices. One can see that the historical prices appearing in Figure 1 are more stable than the actual prices appearing in Figure 3. This stability in the historical time series affected the computations of all parameters, including the forecasting methods under study. Therefore, the forecast prices fluctuate less than the actual prices, which ensures that the forecast prices are closer together.

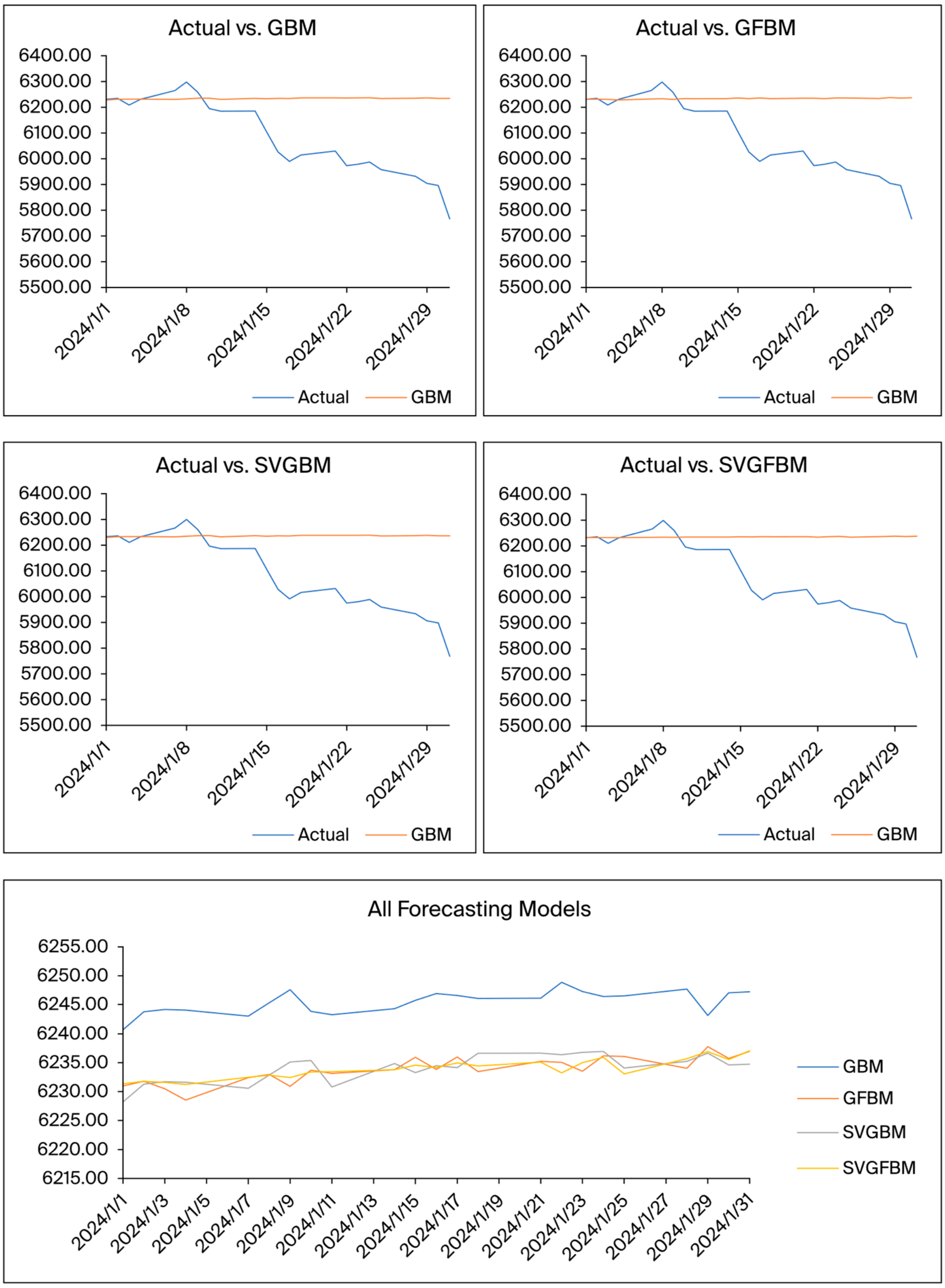


**Figure 3.** Actual prices vs. forecast prices.

## 4. Conclusions

Index price reflects financial stability and economic growth. Therefore, forecasting the future performance of index prices is one of the top tasks of stakeholders and investors. For this task, several models have been presented in the literature. The GBM model is one of the most popular and important forecasting models. Furthermore, a number of models have been developed based on the classical GBM model, which depends on the assumption of the existence (or absence) of memory in time-series financial data in addition to the assumption of volatility (constant or stochastic). To discuss the benefits of these assumptions, this work examined four GBM models, including the classical GBM model (absence of memory and constant volatility), the GFBM model (existence of memory and constant volatility), the SVGBM model (absence of memory and stochastic volatility), and the SVGFBM model (memory and stochastic volatility). These models are described in Section 2 of this manuscript. The examination in this study was conducted by applying these models to forecast the energy sector in the Saudi Stock Exchange Market. Two statistical criteria of error were utilized (MSE and MAPE) to evaluate the performance of each model.

The findings of this empirical examination show that the SVGFBM model achieved the smallest MSE and MAPE and, therefore, the best performance. This result was achieved because of the existence of two sources of randomness with memory ($dB_{H_1}$ and $dB_{H_2}$) and the assumption of stochastic volatility. The GBM model ranked last because of the existence of one source of randomness without memory ($H = 0.5$) and the assumption of constant volatility.

The results demonstrate that GFBM models are more accurate than GBM models for forecasting future stock prices. Furthermore, under the same assumption of memory, the models of stochastic volatility assumption are more accurate than the models of constant volatility assumption. This outcome proved the direct positive benefits of incorporating memory and stochastic volatility together into GBM models.

Moreover, the MSE values for the SVGFBM, GFBM, and SVGBM models were close in number and thus more stable, while those of the GBM model were moderately larger than the others. Generally, according to Lawrence's table of judgment accuracy (Table 2), all the models exhibited a high performance because of the MAPEs < 10%, which indicates that all the models under study can be used as tools for forecasting future index prices of the energy sector in Saudi Arabia.

In general, the empirical results of this study agree with earlier empirical studies, such as those by Abbas and Alhagyan (2022, 2023), Mansour and Ayasrah (2022), Alhagyan and Yassen (2023), Alhagyan (2022), Willinger et al. (1999), Painter (1998), and Rejichi and Aloui (2012). Therefore, under normal circumstances, we encourage investors and traders to invest in Saudi Arabia's energy sector because of its demonstrated predictability and stability.

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