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Proportional Caputo Fractional Differential Inclusions in Banach Spaces

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Abstract: In this work, we introduce the notion of a (weak) proportional Caputo fractional derivative of order $\alpha \in (0,1)$ for a continuous (locally integrable) function $u : [0,\infty) \to E$, where E is a complex Banach space. In our definition, we do not require that the function $u(\cdot)$ is continuously differentiable, which enables us to consider the wellposedness of the corresponding fractional relaxation problems in a much better theoretical way. More precisely, we systematically investigate several new classes of (degenerate) fractional solution operator families connected with the use of this type of fractional derivatives, obeying the multivalued linear approach to the abstract Volterra integro-differential inclusions. The quasi-periodic properties of the proportional fractional integrals as well as the existence and uniqueness of almost periodic-type solutions for various classes of proportional Caputo fractional differential inclusions in Banach spaces are also considered.

Keywords: fractional differential equations; proportional fractional integrals; proportional Caputo fractional derivatives; abstract Volterra integro-differential inclusions; almost periodic-type functions

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1. Introduction and Preliminaries

Fractional calculus and fractional differential equations have received much attention during the last five decades or so. The dynamics of certain processes appearing in physics, biology, chemistry, population dynamics, ecology and pharmacokinetics, e.g., can be adequately modeled with the help of fractional differential equations. The theory of fractional calculus is the theory of derivatives and integrals with non-integer order, unifying and generalizing the concepts of integer differentiation and integration [1–3].

Applications in nonlinear oscillations of earthquakes and the modeling of many phenomena in engineering, biology and physics, such as seepage flow in porous media and fluid dynamic traffic model (see [4,5]) are only few examples emphasizing the importance and applicability of this theory in studying complex dynamical systems. It would be really difficult to summarize here all applications of fractional calculus and fractional differential equations.

The proportional Caputo fractional derivative is a relatively new type of fractional derivative extending the classical Caputo fractional derivative. In 2017, F. Jarad et al. [6] (see also [7]) introduced this notion, which has many advantages when compared with the notion of the classical Caputo fractional derivative. This concept clearly enables one to consider some broader applications in modeling of various phenomena appearing in natural and technical sciences; see, e.g., [8–15].

As already mentioned in the abstract, we introduce here the notion of a (weak) proportional Caputo fractional derivative of order $\alpha \in (0,1)$ for a continuous (locally integrable) function $u : [0,\infty) \to E$, where E is a complex Banach space. The main novelty of our research study lies in the fact that we do not require the continuous differentiability of function function $u(\cdot)$ in our definition. This enables one to analyze the well-posedness of the associated fractional relaxation problems in a much better theoretical way.

Therefore, we provide here a new theoretical concept of the proportional Caputo fractional derivatives. More to the point, we consider various classes of abstract (degenerate) fractional solution operator families connected with the use of proportional Caputo fractional derivatives; in contrast with a great number of previous research studies, we follow the multivalued linear operators approach to the abstract Volterra integro-differential inclusions here. We also analyze the existence and uniqueness of almost periodic-type solutions for various classes of proportional Caputo fractional differential inclusions in Banach spaces as well as the quasi-periodic properties of the proportional fractional integrals (see also [16–20]).

Let us recall that the class of of almost periodic functions was introduced by the Danish mathematician H. Bohr around 1924–1926 and later reconsidered by many other authors (cf. [20–24] for more details on the subject). Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $f : I \to X$ be continuous, where $(X, \| \cdot \|)$ is a complex Banach space. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if and only if $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$. The set consisting of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. It is said that $f(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta(f,\epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f,\epsilon)$.

Let $f : \mathbb{R} \to X$ be continuous. Then, it is said that $f(\cdot)$ is almost automorphic, if and only if, for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a mapping $g : \mathbb{R} \to X$ such that

$$\lim_{n\to\infty} f(t+a_n) = g(t) \text{ and } \lim_{n\to\infty} g(t-a_n) = f(t), \tag{1}$$

pointwise for $t \in \mathbb{R}$. If this is the case, then $f(\cdot)$ and $g(\cdot)$ are bounded but the limit function $g(\cdot)$ is not necessarily continuous on $\mathbb{R}$. If the convergence of limits appearing in (1) is uniform on compact subsets of $\mathbb{R}$, then we say that $f(\cdot)$ is compactly almost automorphic. By Bochner's criterion, any almost periodic function is compactly almost automorphic; the converse statement is not true, in general. We know that an almost automorphic function $f : \mathbb{R} \to X$ is compactly almost automorphic if and only if $f(\cdot)$ is uniformly continuous.

We also need the following generalization of almost periodicity: Let $I = \mathbb{R}$ or $I = [0,\infty)$. Then, a continuous function $f : I \to X$ is said to be uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and $\lim_{n\to+\infty} f(t+\alpha_n) = f(t)$, uniformly in $t \in I$. It is well known that there exists a compactly almost automorphic function $f : \mathbb{R} \to \mathbb{R}$, which is not uniformly recurrent as well as that there exists a bounded, uniformly continuous and uniformly recurrent function $f : \mathbb{R} \to \mathbb{R}$, which is not compactly almost automorphic; for more detail about these classes of almost periodic functions, we refer the reader to the newly published research monograph [25] and references cited therein.

The important generalization of uniform recurrence is Poisson stability: A continuous function $f : I \to X$ is said to be Poisson stable if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and $\lim_{n\to+\infty} f(t+\alpha_n) = f(t)$, uniformly on compact subsets of I. The notion of Poisson stability will be important in our further work (for more details about the subject, we refer the reader to the research monograph [26] by B. A. Shcherbakov, as well as to the research articles [27–29], the forthcoming research article [30] and the list of references cited therein).

Concerning the qualitative analysis of solutions of the ordinary differential equations and the partial differential equations, we also recall that the class of (ω, c)-periodic functions was introduced and investigated by E. Alvarez et al. in [31,32]: Suppose that $\omega > 0$ and $c \in \mathbb{C}\setminus\{0\}$. Then, a continuous function $u : I \to X$ is said to be (ω, c)-periodic [31] if and only if $u(t + \omega) = cu(t)$ for all $t \in I$. If $c = 1$ $[c = -1]$, then we obtain the class of ω-periodic functions [ω-antiperiodic functions].

In [33], the authors investigated the existence and uniqueness of (ω, c)-periodic solutions for semilinear evolution equations $u' = Au + f(t, u)$ in complex Banach spaces, while in [34], the necessary and sufficient conditions for the existence of (ω, c)-periodic solutions to a certain type of impulsive fractional differential equations were considered (see also [25] and references cited therein for more details about (ω, c)-periodic functions and their applications). Define

$$\Phi_{\omega,c} := \{u \in C_b(I : X) ; u(\omega) = cu(0)\},$$

where $C_b(I : X)$ denotes the Banach space of all bounded continuous functions from I into X, equipped with the sup-norm. It is clear that any (ω, c)-periodic function $u : I \to X$ belongs to the space $\Phi_{\omega,c}$ if $|c| \leq 1$ as well as that the converse statement is far from being generally true; furthermore, $\Phi_{\omega,c}$ is a closed linear subspace of $C_b(I : X)$, and therefore, a Banach space itself when equipped with the sup-norm.

For the sequel, we need to recall the following notion from [25] as well: Let $\omega > 0$ and $c \in \mathbb{C}\setminus\{0\}$. Then, a continuous function $f : [0, \infty) \to X$ is said to be (ω, c)-almost periodic [S-asymptotically (ω, c)-periodic] if and only if the function $c^{-\cdot/\omega} f(\cdot)$ is almost periodic $[\lim_{t\to+\infty} \|f(t + \omega) - cf(t)\| = 0]$. If $|c| = 1$, then we have:

$$\lim_{t\to+\infty} \|f(t + \omega) - cf(t)\| = c \lim_{t\to+\infty} \left\|c^{-\frac{t+\omega}{\omega}} f(t + \omega) - c^{-\frac{t}{\omega}} f(t)\right\|,$$

so that a continuous function $f : [0, \infty) \to X$ is S-asymptotically (ω, c)-periodic if and only if the function $c^{-\cdot/\omega} f(\cdot)$ is S-asymptotically ω-periodic, that is, S-asymptotically (ω, c)-periodic with $c = 1$. Furthermore, if $|c| \geq 1$, then we have that the function $c^{-\cdot/\omega} f(\cdot)$ is S-asymptotically ω-periodic if the function $f(\cdot)$ is S-asymptotically (ω, c)-periodic. Although we will not use this fact in the sequel, let us only note that, if $|c| \geq 1$ and a continuous function $f : [0, \infty) \to X$ is both, S-asymptotically (ω, c)-periodic and (ω, c)-almost periodic, then $f(\cdot)$ is necessarily (ω, c)-periodic function; this can be shown with the help of substitution $c^{-(\cdot/\omega)} f(\cdot)$, ([35], Proposition 2) and the corresponding statement with $c = 1$.

The structure and main ideas of this paper can be briefly described as follows. In Section 1.1, we recollect some necessary definitions and results from the theory of multivalued linear operators; we also consider (degenerate) solution operator families subgenerated by multivalued linear operators here. Section 2 investigates the proportional fractional integrals and the proportional Caputo fractional derivatives of vector-valued functions. The main theoretical results are given in Section 3, where we study the abstract proportional fractional differential inclusions in Banach spaces (for simplicity, we will not consider the solution operator families in locally convex spaces here; see [36] for more details about this topic).

Solution operator families for the abstract fractional Cauchy problems $(\mathrm{DFP})^{\zeta}_{\mathbf{R}}$ and $(\mathrm{DFP})^{\zeta}_{\mathbf{L}}$ are investigated in Section 3.1; a few relevant applications to the abstract Volterra integro-differential inclusions are given in Section 3.2. Further on, some results about the existence and uniqueness of (ω, c)-periodic-type solutions for some special classes of the semilinear proportional Caputo fractional differential equations are given in Section 4.

In Section 5.1, we consider the quasi-periodic properties of proportional fractional integrals and observe that the quasi-periodic properties of Riemann–Liouville (Caputo) fractional derivatives established in the research studies [35,37] by I. Area, J. Losada and J. J. Nieto hold for the vector-valued functions. We apply these results in the continuation of

Section 5, where we prove the nonexistence of (ω, c)-periodic solutions of the semilinear fractional Cauchy problem (17) and the nonexistence of Poisson stable-like solutions of the same problem.

The final section of paper is reserved for the final remarks and observations about the introduced notion and obtained results. In this paper, we will consider the proportional fractional integrals, the proportional Caputo fractional derivatives of order $\alpha \in (0,1)$, and the corresponding abstract fractional relaxation inclusions with the proportional Caputo fractional derivatives, only. The proportional Caputo fractional derivatives of order $\alpha > 1$, and the corresponding abstract fractional oscillation inclusions will be considered elsewhere.

Before proceeding to Section 1.1, we briefly explain the notation and terminology used in this paper. By $(E, \|\cdot\|)$ and $(X, \|\cdot\|)$ we denote two complex Banach spaces (since no confusion seems likely, we denote the norms in these spaces by the same symbols); $I = [0,\infty)$, $\omega > 0$ and $c \in \mathbb{C}\setminus\{0\}$. By $L(E,X)$ we denote the space consisting of all continuous linear mappings from E into X; $L(E) \equiv L(E,E)$. If A is a closed linear operator acting on X, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively.

Given $\alpha \in (0,\pi]$ in advance, set $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$. The Gamma function is denoted by $\Gamma(\cdot)$, and the principal branch is always used to take the powers; the convolution-like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. Put $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ and $0^\zeta := 0$ ($\zeta > 0$, $t > 0$). The symbol $\operatorname{supp}(f)$ denotes the support of a function $f(\cdot)$. We employ the following condition on a vector-valued function $k(\cdot)$:

(P1) $k(\cdot)$ is Laplace transformable, i.e., $k \in L^1_{loc}([0,\infty) : X)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k(t))(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t}k(t)\,dt := \int_0^\infty e^{-\lambda t}k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put $\operatorname{abs}(k) := \inf\{\Re\lambda : \tilde{k}(\lambda) \text{ exists}\}$.

Assume that $\alpha \in (0,1)$ and $T \in (0,\infty]$. Then, the Riemann–Liouville fractional integral J_t^α of order α is defined by

$$J_t^\alpha f(t) := \big(g_\alpha * f\big)(t), \quad f \in L^1([0,T) : X),\ t \in [0,T).$$

The Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$ is defined for those continuous functions $u \in C([0,T) : X)$ for which $g_{1-\alpha} * (u(\cdot) - u(0)) \in C^1([0,T) : X)$, by

$$\mathbf{D}_t^\alpha u(t) := \frac{d}{dt}\Big[g_{1-\alpha} * \big(u(\cdot) - u(0)\big)\Big], \quad t \in [0,T).$$

In a slightly weakened concept, we define the Caputo fractional derivative $\mathbf{D}_{t,w}^\alpha u(t)$ for those functions $u : [0,T) \to X$ for which $u_{|(0,T)}(\cdot) \in C((0,T) : X)$, $u(\cdot) - u(0) \in L^1((0,T) : X)$ and $g_{1-\alpha} * (u(\cdot) - u(0)) \in W^{1,1}((0,T) : X)$, by

$$\mathbf{D}_{t,w}^\alpha u(t) := \frac{d}{dt}\Big[g_{1-\alpha} * \big(u(\cdot) - u(0)\big)\Big], \quad t \in (0,T).$$

Here, $W^{1,1}((0,T) : X)$ denotes the usual Sobolev space of order 1; see, e.g., [38] and references cited therein. Note that, for a given function $u_{|(0,T)}(\cdot) \in C((0,T) : X)$, there exists only one value $u(0)$ such that $g_{1-\alpha} * (u(\cdot) - u(0)) \in W^{1,1}((0,T) : X)$.

Remark 1. *It is clear that $\mathbf{D}_t^\alpha[Const.] = 0$ for any $\alpha \in (0,1)$ so that we must replace the word "nonzero" in the formulation of ([35], Corollary 2) with the word "nonconstant" in order to retain its validity. We will use this fact later on.*

1.1. Multivalued Linear Operators and Solution Operator Families Subgenerated by Them

In this subsection, we provide a brief overview of definitions and results about multivalued linear operators and (degenerate) (a,k)-regularized C-resolvent families subgenerated by them. For more details about the subject, we refer the reader to the research monographs [39] by R. Cross, [40] by A. Favini, A. Yagi and [36] by M. Kostić. A multivalued mapping $\mathcal{A} : E \to P(E)$ is said to be a multivalued linear operator (MLO in E, or simply, MLO) if and if the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear submanifold of E;
(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x, y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

We know that, for every $x, y \in D(\mathcal{A})$ and for every $\lambda, \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. Furthermore, $\mathcal{A}0$ is a linear manifold in E and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Define $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1}0 := N(\mathcal{A}) := \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$. The inverse $\mathcal{A}^{-1}$ is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It can be proven simply that $\mathcal{A}^{-1}$ is an MLO in E, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$.

Suppose now that $\mathcal{A}$, $\mathcal{B}$ are two MLOs in E. Then, its sum $\mathcal{A} + \mathcal{B}$ is defined by $D(\mathcal{A} + \mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A} + \mathcal{B})$. It is clear that $\mathcal{A} + \mathcal{B}$ is an MLO in E. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. We have that $\mathcal{B}\mathcal{A}$ is an MLO in E and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The inclusion $\mathcal{A} \subseteq \mathcal{B}$ means that $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. The scalar multiplication of an MLO $\mathcal{A}$ with a complex number number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$.

We say that an MLO operator $\mathcal{A}$ is closed if and only if for any nets (x_τ) in $D(\mathcal{A})$ and (y_τ) in E such that $y_\tau \in \mathcal{A}x_\tau$ for all $\tau \in I$ we have that the preassumptions $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} y_\tau = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

We need the following lemma [36]:

Lemma 1. *Suppose that $\mathcal{A}$ is a closed* MLO *in E, Ω is a locally compact and separable metric space, as well as that μ is a locally finite Borel measure defined on Ω. Let $f : \Omega \to E$ and $g : \Omega \to E$ be μ-integrable and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then, $\int_\Omega f\,d\mu \in D(\mathcal{A})$ and $\int_\Omega g\,d\mu \in \mathcal{A}\int_\Omega f\,d\mu$.*

Let $\mathcal{A}$ be an MLO in E, $C \in L(E)$ be injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;
(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on E.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$), where I denotes the identity operator on E. The basic properties of C-resolvent sets of single-valued linear operators continue to hold [36].

Of concern is the following abstract degenerate Volterra inclusion:

$$\mathcal{B}u(t) \subseteq \mathcal{A}\int_0^t a(t-s)u(s)ds + \mathcal{F}(t), \quad t \in [0,\tau), \tag{2}$$

where $0 < \tau \leq \infty$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{F}\colon [0,\tau) \to P(E)$, and $\mathcal{A}\colon X \to P(E)$, $\mathcal{B}\colon X \to P(E)$ are two given mappings (possibly non-linear). We need the following notion:

Definition 1 (cf. [36], Definition 3.1.1(i))**.**

(i) *A function $u \in C([0,\tau) : X)$ is said to be a pre-solution of* (2) *if and only if $(a * u)(t) \in D(\mathcal{A})$ and $u(t) \in D(\mathcal{B})$ for $t \in [0,\tau)$, as well as* (2) *holds.*
(ii) *A solution of* (2) *is any pre-solution $u(\cdot)$ of* (2) *satisfying additionally that there exist functions $u_\mathcal{B} \in C([0,\tau) : E)$ and $u_{a,\mathcal{A}} \in C([0,\tau) : E)$ such that $u_\mathcal{B}(t) \in \mathcal{B}u(t)$ and $u_{a,\mathcal{A}}(t) \in \mathcal{A}\int_0^t a(t-s)u(s)ds$ for $t \in [0,\tau)$, as well as*

$$u_\mathcal{B}(t) \in u_{a,\mathcal{A}}(t) + \mathcal{F}(t), \quad t \in [0,\tau).$$

(iii) *A strong solution of* (2) *is any function* $u \in C([0,\tau) : X)$ *satisfying that there exist two continuous functions* $u_{\mathcal{B}} \in C([0,\tau) : E)$ *and* $u_{\mathcal{A}} \in C([0,\tau) : E)$ *such that* $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$, $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ *for all* $t \in [0,\tau)$, *and*

$$u_{\mathcal{B}}(t) \in (a * u_{\mathcal{A}})(t) + \mathcal{F}(t), \quad t \in [0,\tau).$$

In the remainder of this subsection, we will analyze multivalued linear operators as subgenerators of (a,k)-regularized (C_1,C_2)-existence and uniqueness families and (a,k)-regularized C-resolvent families. Unless specified otherwise, we assume that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : E \to P(E)$ is an MLO, $C_1 \in L(X,E)$, $C_2 \in L(E)$ is injective, $C \in L(E)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$.

We need the following notion (see., e.g., [36], Definition 3.2.1, Definition 3.2.2):

Definition 2.

(i) *It is said that* $\mathcal{A}$ *is a subgenerator of a (local, if* $\tau < \infty$*) mild* (a,k)*-regularized* (C_1,C_2)*-existence and uniqueness family* $(R_1(t),R_2(t))_{t\in[0,\tau)} \subseteq L(X,E) \times L(E)$ *if and only if the mappings* $t \mapsto R_1(t)y$, $t \geq 0$ *and* $t \mapsto R_2(t)x$, $t \in [0,\tau)$ *are continuous for every fixed* $x \in E$ *and* $y \in X$, *as well as the following conditions hold:*

$$\left(\int_0^t a(t-s)R_1(s)y\,ds, R_1(t)y - k(t)C_1y\right) \in \mathcal{A},\ t \in [0,\tau),\ y \in X \text{ and} \tag{3}$$

$$\int_0^t a(t-s)R_2(s)y\,ds = R_2(t)x - k(t)C_2x, \text{ whenever } t \in [0,\tau) \text{ and } (x,y) \in \mathcal{A}. \tag{4}$$

(ii) *Let* $(R_1(t))_{t\in[0,\tau)} \subseteq L(X,E)$ *be strongly continuous. Then, it is said that* $\mathcal{A}$ *is a subgenerator of a (local, if* $\tau < \infty$*) mild* (a,k)*-regularized* C_1*-existence family* $(R_1(t))_{t\in[0,\tau)}$ *if and only if* (3) *holds.*

(iii) *Let* $(R_2(t))_{t\in[0,\tau)} \subseteq L(E)$ *be strongly continuous. Then, it is said that* $\mathcal{A}$ *is a subgenerator of a (local, if* $\tau < \infty$*) mild* (a,k)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\in[0,\tau)}$ *if and only if* (4) *holds.*

Definition 3. *Suppose that* $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : E \to P(E)$ *is an MLO,* $C \in L(E)$ *is injective and* $C\mathcal{A} \subseteq \mathcal{A}C$. *Then, it is said that a strongly continuous operator family* $(R(t))_{t\in[0,\tau)} \subseteq L(E)$ *is an* (a,k)*-regularized* C*-resolvent family with a subgenerator* $\mathcal{A}$ *if and only if* $(R(t))_{t\in[0,\tau)}$ *is a mild* (a,k)*-regularized* C*-uniqueness family having* $\mathcal{A}$ *as subgenerator,* $R(t)C = CR(t)$ *and* $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \in [0,\tau))$.

If $\tau = \infty$, then $(R(t))_{t\geq 0}$ is said to be exponentially bounded (bounded) if and only if there exists $\omega \in \mathbb{R}$ $(\omega = 0)$ such that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is bounded; the infimum of such numbers is said to be the exponential type of $(R(t))_{t\geq 0}$. The above notion can be simply understood for the classes of mild (a,k)-regularized C_1-existence families and mild (a,k)-regularized C_2-uniqueness families.

The integral generator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)}$) is defined by

$$\mathcal{A}_{int} := \left\{(x,y) \in X \times X : R_2(t)x - k(t)C_2x = \int_0^t a(t-s)R_2(s)y\,ds,\ t \in [0,\tau)\right\};$$

we define the integral generator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ in the same way.

For simplicity, we will assume that any (a,k)-regularized C-resolvent family considered below is likwise a mild (a,k)-regularized C-existence family (subgenerated by $\mathcal{A}$). We refer the reader to [36,41] for several simple conditions ensuring this property.

2. Proportional Fractional Integrals and Proportional Caputo Fractional Derivatives

First, we recall the definition of proportional fractional integral of a locally integrable function $u : [0,\infty) \rightarrow X$ (see [6]):

$$\left({}_0I^{\alpha,\zeta}u\right)(t) := \frac{1}{\zeta^{\alpha}\Gamma(\alpha)} \int\limits_0^t e^{\frac{\zeta-1}{\zeta}(t-s)}(t-s)^{\alpha-1}u(s)\,ds, \quad t \geq 0,\ \alpha \geq 0,\ \zeta \in (0,1].$$

If the function $u : [0,\infty) \rightarrow X$ is differentiable and its first derivative is locally integrable, then we define its proportional Caputo fractional derivative by

$$\left({}_0^CD^{\alpha,\zeta}u\right)(t) := \left({}_0I^{1-\alpha,\zeta}\left(D^{1,\zeta}u\right)\right)(t)$$

$$:= \frac{1}{\zeta^{1-\alpha}\Gamma(1-\alpha)} \int\limits_0^t e^{\frac{\zeta-1}{\zeta}(t-s)}(t-s)^{-\alpha}\left(D^{1,\zeta}u\right)(s)\,ds,$$
$$\text{for} \quad t \geq 0,\ \alpha \in (0,1),\ \zeta \in (0,1],$$

where
$$\left(D^{1,\zeta}u\right)(t) := \left(D^{\zeta}u\right)(t) := (1-\zeta)u(t) + \zeta u'(t).$$

For $\zeta = 1$, the proportional Caputo fractional derivative is reduced to the classical Caputo fractional derivative. Additionally, for $\zeta \in (0,1]$ and $\alpha \in (0,1)$, we note that

$$\left({}_0I^{\alpha,\zeta}\left({}_0^CD^{\alpha,\zeta}u\right)\right)(t) = u(t) - u(0)e^{\frac{\zeta-1}{\zeta}t}, \quad t \geq 0 \tag{5}$$

and

$$\left({}_0^CD^{\alpha,\zeta}\left({}_0I^{\alpha,\zeta}u\right)\right)(t) = u(t), \quad t \geq 0. \tag{6}$$

For our further work, it will be important to observe that the proportional Caputo fractional derivatives of order α can be defined even for a continuous (locally integrable) function $u : [0,\infty) \rightarrow X$ (for example, it is not so satisfactory to consider the well-posedness of the fractional relaxation problem $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ below for the continuously differentiable functions; see Definition 5 below). In actual fact, if the function $u : [0,\infty) \rightarrow X$ is differentiable and its first derivative is locally integrable, then we have:

$$\left({}_0^CD^{\alpha,\zeta}u\right)(t) := \zeta^{\alpha-1}(1-\zeta) \int\limits_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} g_{1-\alpha}(t-s)u(s)\,ds$$
$$+ \zeta^{\alpha} \int\limits_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} g_{1-\alpha}(t-s)u'(s)\,ds := I(t) + II(t), \quad t \geq 0.$$

The term $I(t)$ is clearly definable for any continuous (locally integrable) function $u : [0,\infty) \rightarrow X$. Concerning the term $II(t)$, we manipulate as follows: Clearly,

$$e^{\frac{1-\zeta}{\zeta}t}II(t) = \zeta^{\alpha}\left(g_{1-\alpha} * e^{\frac{1-\zeta}{\zeta}\cdot}u'(\cdot)\right)(t), \quad t \geq 0,$$

so that the partial integration implies

$$\Big(g_\alpha * e^{\frac{1-\zeta}{\zeta}\cdot} II(\cdot)\Big)(t) = \zeta^\alpha \Big(g_1 * e^{\frac{1-\zeta}{\zeta}\cdot} u'(\cdot)\Big)(t)$$
$$= \zeta^\alpha \left(e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds\right)(t), \quad t \geq 0.$$

Convoluting with $g_{1-\alpha}(\cdot)$, we find that, for every $t \geq 0$,

$$\Big(g_1 * e^{\frac{1-\zeta}{\zeta}\cdot} II(\cdot)\Big)(t) = \zeta^\alpha \left[g_{1-\alpha} * \left(e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds\right)\right](t),$$

so that

$$II(t) = \zeta^\alpha e^{\frac{\zeta-1}{\zeta}t} \frac{d}{dt}\left[g_{1-\alpha} * \left(e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds\right)\right](t), \quad t \geq 0. \tag{7}$$

Although Equation (7) can be used to provide the definition of the proportional Caputo fractional derivatives of order α for any locally integrable function $u(\cdot)$, we will restrict ourselves to the following notion:

Definition 4. *Let $T \in (0, \infty]$.*

(i) *Suppose that $u \in C([0,T) : X)$. The proportional Caputo fractional derivative $\mathbf{D}_t^{\alpha,\zeta} u(t)$ is defined provided $g_{1-\alpha} * (e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds) \in C^1([0,T) : X)$, by*

$$\mathbf{D}_t^{\alpha,\zeta} u(t) := \zeta^{\alpha-1}(1-\zeta)\int_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} g_{1-\alpha}(t-s) u(s)\,ds$$
$$+ \zeta^\alpha e^{\frac{\zeta-1}{\zeta}t} \frac{d}{dt}\left[g_{1-\alpha} * \left(e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds\right)\right](t), \quad t \in [0,T).$$

(ii) *We define the proportional Caputo fractional derivative $\mathbf{D}_{t,w}^{\alpha,\zeta} u(t)$ for those functions $u : [0,T) \to X$ for which $u_{|(0,T)}(\cdot) \in C((0,T) : X)$, $e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds \in L^1((0,T) : X)$ and $g_{1-\alpha} * (e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds) \in W^{1,1}((0,T) : X)$, by*

$$\mathbf{D}_{t,w}^{\alpha,\zeta} u(t) := \zeta^{\alpha-1}(1-\zeta)\int_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} g_{1-\alpha}(t-s) u(s)\,ds$$
$$+ \zeta^\alpha e^{\frac{\zeta-1}{\zeta}t} \frac{d}{dt}\left[g_{1-\alpha} * \left(e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds\right)\right](t), \quad t \in (0,T).$$

For $\zeta = 1$, our proportional Caputo fractional derivatives reduce to $\mathbf{D}_t^\alpha u(t)$ and $\mathbf{D}_{t,w}^\alpha u(t)$. Furthermore, suppose that $1 > \alpha > \beta > 0$; then, immediately from Definition 4, it follows that the existence of fractional derivative $\mathbf{D}_t^{\alpha,\zeta} u(t)$ ($\mathbf{D}_{t,w}^{\alpha,\zeta} u(t)$) implies the existence of fractional derivative $\mathbf{D}_t^{\beta,\zeta} u(t)$ ($\mathbf{D}_{t,w}^{\beta,\zeta} u(t)$).

Remark 2. *Denote $A(t) = [g_{1-\alpha} * (e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds)](t)$, $t \in [0,T)$ and $B(t) = [g_{1-\alpha} * (e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot) - u(0))](t)$, $t \in [0,T)$. Since $\int_0^t e^{\frac{1-\zeta}{\zeta}s} u(s)\,ds = (g_1 * e^{\frac{1-\zeta}{\zeta}\cdot} u(\cdot))(t)$, $t \geq 0$, we have the following: Suppose that $u \in C([0,T) : X)$, resp. $u \in L^1([0,T) : X)$. Then,*

$A \in C^1([0,T) : X)$, resp. $A \in W^{1,1}((0,T) : X)$, if and only if $B \in C^1([0,T) : X)$, resp. $B \in W^{1,1}((0,T) : X)$. It is also worth noting that, for a given function $u_{|(0,T)}(\cdot) \in C((0,T) : X)$, there exists only one value $u(0)$ such that $g_{1-\alpha} * (e^{\frac{1-\zeta}{\zeta}\cdot}u(\cdot) - u(0) - \frac{1-\zeta}{\zeta}\int_0^{\cdot} e^{\frac{1-\zeta}{\zeta}s}u(s)\,ds) \in W^{1,1}((0,T) : X)$.

It is worth noting that Definition 4(ii) can be used to compute the value of the fractional derivative $\mathbf{D}_{t,w}^{\alpha,\zeta} g_a(t)$ under certain assumptions:

Example 1. *Suppose that $0 < \alpha < a < 1$ and $0 < \zeta < 1$. Define $u(t) := g_a(t)$ for $t > 0$ and $u(0) := 0$. In order to compute $X(t) = (d/dt)[g_{1-\alpha} * e^{\frac{1-\zeta}{\zeta}\cdot}u(\cdot)](t)$, $t > 0$, we integrate this equality, convolve the obtained equality with $g_\alpha(t)$ and differentiate the obtained equality after this; it follows that $e^{\frac{1-\zeta}{\zeta}t}g_a(t) = (g_\alpha * X)(t)$, $t > 0$. Applying the Laplace transform, we obtain*

$$\tilde{X}(\lambda) = \frac{\lambda^\alpha}{(\lambda + \frac{\zeta-1}{\zeta})^a}, \quad \Re\lambda > \frac{1-\zeta}{\zeta},$$

so that $X(t)$ is locally integrable on $[0,\infty)$ and, more precisely,

$$X(t) = t^{a-\alpha-1}\mathcal{E}_{1,a-\alpha}^{a}(-at), \quad t > 0,$$

where

$$\mathcal{E}_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)\Gamma(\alpha k+\beta)}\frac{z^k}{k!}, \quad z \in \mathbb{C}$$

is the generalized Mittag–Leffler function [42]. Therefore,

$$\begin{aligned}\mathbf{D}_{t,w}^{\alpha,\zeta}u(t) &= \zeta^{\alpha-1}(1-\zeta)\int_0^t e^{\frac{\zeta-1}{\zeta}(t-s)}g_{1-\alpha}(t-s)g_a(s)\,ds \\ &+ \zeta^\alpha e^{\frac{\zeta-1}{\zeta}t}t^{a-\alpha-1}\mathcal{E}_{1,a-\alpha}^{a}(-at) - \zeta^\alpha e^{\frac{\zeta-1}{\zeta}t}\frac{1-\zeta}{\zeta}\Big(g_{1-\alpha} * e^{\frac{1-\zeta}{\zeta}\cdot}g_a(\cdot)\Big)(t), \quad t > 0.\end{aligned}$$

It is predictable that the equalities (5) and (6) continue to hold in our new framework:

Proposition 1. *Suppose that $u \in C([0,T) : X)$, resp. $u \in L^1_{loc}([0,T) : X)$. Then, the following holds:*

(i) If $\mathbf{D}_t^{\alpha,\zeta}u(t)$, resp. $\mathbf{D}_{t,w}^{\alpha,\zeta}u(t)$ is well-defined, then we have

$$\left[{}_0I^{\alpha,\zeta}\left(\mathbf{D}_{t,w}^{\alpha,\zeta}u\right)\right](t) = u(t) - u(0)e^{\frac{\zeta-1}{\zeta}t}, \quad \text{for } t \in [0,T), \quad \text{resp., for a.e. } t \in (0,T). \quad (8)$$

(ii) We have

$$\left[\mathbf{D}_{t,w}^{\alpha,\zeta}\left({}_0I^{\alpha,\zeta}u\right)\right](t) = u(t), \quad \text{for } t \in [0,T), \quad \text{resp., for a.e. } t \in (0,T). \quad (9)$$

Proof. The proofs of both statements follows from the relative simple but tedious computations; for the sake of completeness, we will prove only (ii). By our definitions, we need to show that

$$\zeta^{\alpha-1}(1-\zeta)\Big(e^{\frac{\zeta-1}{\zeta}\cdot}g_{1-\alpha}(\cdot)*\zeta^{-\alpha}e^{\frac{\zeta-1}{\zeta}\cdot}g_{\alpha}(\cdot)*u\Big)(t)$$
$$+\zeta^{\alpha}e^{\frac{\zeta-1}{\zeta}t}\frac{d}{dt}\Bigg[g_{1-\alpha}*\Big[\Big(e^{\frac{1-\zeta}{\zeta}\cdot}\zeta^{-\alpha}e^{\frac{\zeta-1}{\zeta}\cdot}g_{\alpha}(\cdot)*u\Big)$$
$$-\frac{1-\zeta}{\zeta}\Big(g_1*\Big(e^{\frac{1-\zeta}{\zeta}\cdot}\zeta^{-\alpha}e^{\frac{\zeta-1}{\zeta}\cdot}g_{\alpha}(\cdot)*u\Big)\Big)\Big]\Bigg](t)=u(t),$$

for $t\in[0,T)$, resp., for a.e. $t\in(0,T)$. Since $(g_{1-\alpha}*g_\alpha)(t)=g_1(t)$, $t>0$, it can be proven simply that

$$\zeta^{\alpha-1}(1-\zeta)\Big(e^{\frac{\zeta-1}{\zeta}\cdot}g_{1-\alpha}(\cdot)*\zeta^{-\alpha}e^{\frac{\zeta-1}{\zeta}\cdot}g_{\alpha}(\cdot)*u\Big)(t)=\frac{1-\zeta}{\zeta}\Big(e^{\frac{\zeta-1}{\zeta}\cdot}*u\Big)(t),$$

for $t\in[0,T)$, resp. for a.e. $t\in(0,T)$. Since

$$e^{\frac{1-\zeta}{\zeta}t}\Big(e^{\frac{\zeta-1}{\zeta}\cdot}g_{\alpha}(\cdot)*u\Big)(t)=\Big(g_\alpha*\big[e^{\frac{1-\zeta}{\zeta}\cdot}u(\cdot)\big]\Big)(t),$$

for $t\in[0,T)$, resp. for a.e. $t\in(0,T)$, the previous equality and a simple calculation shows that we need to deduce that

$$\Big(e^{\frac{\zeta-1}{\zeta}\cdot}*u\Big)(t)=e^{\frac{\zeta-1}{\zeta}t}\Big(e^{\frac{1-\zeta}{\zeta}\cdot}*u\Big)(t),$$

for $t\in[0,T)$, resp. for a.e. $t\in(0,T)$. However, this is a trivial equality. □

Using the operational properties of the Laplace transform (see, e.g., [43], Section 1.6, p. 36), we can simply prove that the Laplace transform of the proportional Caputo fractional derivative $\mathbf{D}_{t,w}^{\alpha,\zeta}u(t)$ can be computed by

$$\widetilde{\mathbf{D}_{t,w}^{\alpha,\zeta}u}(\lambda)=\zeta^{\alpha-1}(1-\zeta)\left(\lambda+\frac{1-\zeta}{\zeta}\right)^{\alpha-1}\tilde{u}(\lambda)$$
$$+\zeta^{\alpha}\left(\lambda+\frac{1-\zeta}{\zeta}\right)^{\alpha}\left[\tilde{u}(\lambda)-\frac{u(0)}{\lambda+\frac{1-\zeta}{\zeta}}-\frac{1-\zeta}{\zeta}\frac{\tilde{u}(\lambda)}{\lambda+\frac{1-\zeta}{\zeta}}\right],\ \Re\lambda>\max(0,\mathrm{abs}(u)), \tag{10}$$

provided that the function $u(t)$ satisfies (P1).

Remark 3. *Observe that we cannot generally expect the validity of Equation (10) for $\Re\lambda>\max(\frac{\zeta-1}{\zeta},abs(u))$ since we need to apply ([43], Corollary 1.6.6) here.*

3. Abstract Proportional Caputo Fractional Differential Inclusions

Of concern are the following proportional Caputo fractional differential inclusions:

$$\begin{cases}(\mathrm{DFP})_{\mathbf{R}}^{\zeta}:\mathbf{D}_t^{\alpha,\zeta}Bu(t)\in\mathcal{A}u(t)+\mathcal{F}(t), & \alpha\in(0,1),\ \zeta\in(0,1],\ t\geq 0,\\ Bu(0)=Bu_0,\end{cases}$$

and

$$\begin{cases}(\mathrm{DFP})_{\mathbf{L}}^{\zeta}:\mathcal{B}\mathbf{D}_t^{\alpha,\zeta}u(t)\in\mathcal{A}u(t)+\mathcal{F}(t), & \alpha\in(0,1),\ \zeta\in(0,1],\ t\geq 0,\\ u(0)=u_0,\end{cases}$$

where $\mathcal{F}\colon [0,\infty) \to P(E)$, $\mathcal{A}\colon X \to P(E)$ and $\mathcal{B}\colon X \to P(E)$ are given mappings (possibly non-linear), and $B : D(B) \subseteq X \to E$ is a single-valued operator. In the following definition, we extend the notion introduced recently in ([36], Definition 3.1.1(ii)-(iii)), where we considered the case $\zeta = 1$:

Definition 5.

(i) (a) *By a p-solution of* $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$*, we mean any X-valued function* $t \mapsto u(t)$, $t \geq 0$ *such that the term* $t \mapsto \mathbf{D}_t^{\alpha,\zeta} Bu(t)$, $t \geq 0$ *is well-defined,* $u(t) \in D(\mathcal{A})$ *for* $t \geq 0$*, and the requirements of* $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ *hold.*

(b) *A pre-solution of* $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ *is any p-solution of* $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ *that is continuous for* $t \geq 0$.

(c) *A solution of* $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ *is any pre-solution* $u(\cdot)$ *of* $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ *such that there exists a function* $u_{\mathcal{A}} \in C([0,\infty) : E)$ *with* $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ *for* $t \geq 0$*, and* $\mathbf{D}_t^{\alpha,\zeta} Bu(t) \in u_{\mathcal{A}}(t) + \mathcal{F}(t)$, $t \geq 0$.

(ii) (a) *By a pre-solution of* $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$*, we mean any continuous X-valued function* $t \mapsto u(t)$, $t \geq 0$ *such that the term* $t \mapsto \mathbf{D}_t^{\alpha,\zeta} u(t)$, $t \geq 0$ *is well defined and continuous, as well as that* $\mathbf{D}_t^{\alpha,\zeta} u(t) \in D(\mathcal{B})$ *and* $u(t) \in D(\mathcal{A})$ *for* $t \geq 0$*, and* $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$ *holds.*

(b) *A solution of* $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$ *is any pre-solution* $u(\cdot)$ *of* $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$ *such that there exist functions* $u_{\alpha,\mathcal{B}} \in C([0,\infty) : E)$ *and* $u_{\mathcal{A}} \in C([0,\infty) : E)$ *such that* $u_{\alpha,\mathcal{B}}(t) \in \mathcal{B}\mathbf{D}_t^{\alpha,\zeta} u(t)$ *and* $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ *for* $t \geq 0$*, as well as that* $u_{\alpha,\mathcal{B}}(t) \in u_{\mathcal{A}}(t) + \mathcal{F}(t)$, $t \geq 0$.

Similarly as above, we can introduce the notion of a (pre-)solution of the problems $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ and $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$ on any finite interval $[0,\tau)$ or $[0,\tau]$, where $0 < \tau < \infty$. We assume henceforth that $\mathcal{A}$ and $\mathcal{B}$ are multivalued linear operators. Before proceeding further, we will only notice that we cannot consider the abstract Cauchy problems $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ or $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$ in full generality by passing to the multivalued linear operators $\mathcal{B}^{-1}\mathcal{A}$ or $\mathcal{A}\mathcal{B}^{-1}$ (see also ([36], Remark 3.1.2) for more details concerning this issue with $\zeta = 1$) as well as that we will revisit the Ljubich's uniqueness criterium ([36], Theorem 3.1.6) for the abstract Cauchy problems with proportional Caputo fractional derivatives elsewhere.

3.1. Solution Operator Families for $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ *and* $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$

In this subsection, we analyze various types of solution operator families for the abstract fractional Cauchy problems $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ and $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$ with $\mathcal{B} = B = I$. For the beginning, let us consider the abstract proportional Caputo inclusion $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ with the function $\mathcal{F}(t) = f(t)$ being single-valued, $X = E$ and the initial value u_0 replaced therein with the initial value Cu_0, where $C \in L(E)$ is injective. Applying (9) and Lemma 1, we find

$$u(t) - e^{\frac{\zeta-1}{\zeta}t} Cu_0 \in \zeta^{-\alpha}\mathcal{A}\int_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} g_\alpha(t-s)u(s)\,ds + \zeta^{-\alpha}\int_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} g_\alpha(t-s)f(s)\,ds, \quad t \geq 0.$$

If $f \equiv 0$, the above justifies the introduction of the following solution operator families for $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$:

Definition 6. *Suppose that* $a(t)$ *and* $k(t)$ *are given by*

$$a(t) := \zeta^{-\alpha} e^{\frac{\zeta-1}{\zeta}t} g_\alpha(t), \quad t > 0 \quad \text{and} \quad k(t) := \Big(e^{\frac{\zeta-1}{\zeta}\cdot} * k_0\Big)(t), \quad t \geq 0, \tag{11}$$

where $k_0(t)$ *is the Dirac delta distribution* $\delta(t)$ *or* $k_0 \in L^1_{loc}([0,\infty))$ *[recall that* $\tilde{\delta} = 1$*]. Then, a mild* (α,ζ,k_0,C_1)*-existence family* $(R_1(t))_{t\in[0,\tau)} \subseteq L(X,E)$ *(a mild* (α,ζ,k_0,C_2)*-uniqueness family* $(R_2(t))_{t\in[0,\tau)} \subseteq L(E)$*; an* (α,ζ,k_0,C)*-resolvent family* $(R(t))_{t\in[0,\tau)} \subseteq L(E)$*) subgener-*

ated by $\mathcal{A}$ is nothing else but a mild (a,k)-regularized C_1-existence family subgenerated by $\mathcal{A}$ (mild (a,k)-regularized C_2-uniqueness family subgenerated by $\mathcal{A}$; (a,k)-regularized C-resolvent family subgenerated by $\mathcal{A}$). The integral generator of a mild (α,ζ,k_0,C_2)-uniqueness family $(R_2(t))_{t\in[0,\tau)} \subseteq L(E)$ [an (α,ζ,k_0,C)-resolvent family $(R(t))_{t\in[0,\tau)} \subseteq L(E)$] is defined to be the integral generator of the corresponding mild (a,k)-regularized C_2-uniqueness family [(a,k)-regularized C-resolvent family].

Observe here that the Titchmarsh convolution theorem yields that $k(t)$ is not identically equal to the zero function as well as that a simple argumentation shows that the function $k(t)$ is continuous for $t \geq 0$. In Definition 6, we assume that the operators C and C_2 are injective; for more details about the situation in which some of these operators is possible non-injective, we refer the reader to ([36], SubSection 3.2.2).

Immediately from Definition 6, it follows that we can apply ([36], Proposition 3.2.3, Proposition 3.2.8, Theorem 3.2.9) and Equation ([36], (274)) to deduce several important structural properties of the introduced solution operator families for the problems $(\mathrm{DFP})^{\zeta}_{\mathbf{R}}$ and $(\mathrm{DFP})^{\zeta}_{\mathbf{L}}$; the properties of subgenerators of solution operator families for these problems can be clarified following the corresponding analysis from ([36], Section 3.2). An application of ([36], Proposition 3.2.13) is possible provided that $k_0(t)$ is the Dirac delta distribution; we will only state here the following particular consequences of ([36], Theorem 3.2.4, Theorem 3.2.5):

Theorem 1. *Suppose $\mathcal{A}$ is a closed* MLO *in X, $C_1 \in L(X,E)$, $C_2 \in L(E)$, C_2 is injective, the kernels $a(t)$ and $k(t)$ are given through (11) and $\omega \geq \max(0, \mathrm{abs}(|k|))$.*

(i) Let $(R_1(t))_{t\geq 0}$ be strongly continuous and let the family $\{e^{-\omega t}R_1(t) : t \geq 0\}$ be equicontinuous. Then, $(R_1(t))_{t\geq 0}$ is a mild (α,ζ,k_0,C_1)-existence family with a subgenerator $\mathcal{A}$ if and only if for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\widetilde{k_0}(\lambda) \neq 0$, we have $R(C_1) \subseteq R(I - \zeta^{-\alpha}(\lambda - ((\zeta-1)/\zeta))^{-\alpha}\mathcal{A})$ and

$$\widetilde{k_0}(\lambda)\frac{C_1 y}{\lambda - \frac{\zeta-1}{\zeta}} \in \left(I - \zeta^{-\alpha}\left(\lambda - \frac{\zeta-1}{\zeta}\right)^{-\alpha}\mathcal{A}\right)\int_0^{\infty} e^{-\lambda t}R_1(t)y\,dt, \quad y \in X.$$

(ii) Let $(R_2(t))_{t\geq 0}$ be strongly continuous and let the family $\{e^{-\omega t}R_2(t) : t \geq 0\}$ be equicontinuous. Then, $(R_2(t))_{t\geq 0}$ is a mild (α,ζ,k_0,C_1)-uniqueness family with a subgenerator $\mathcal{A}$ if and only if for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\widetilde{k_0}(\lambda) \neq 0$, the operator $I - \zeta^{-\alpha}(\lambda - ((\zeta-1)/\zeta))^{-\alpha}\mathcal{A}$ is injective and

$$\widetilde{k_0}(\lambda)\frac{C_2 x}{\lambda - \frac{\zeta-1}{\zeta}} = \int_0^{\infty} e^{-\lambda t}[R_2(t)x - (a * R_2)(t)y]dt, \text{ whenever } (x,y) \in \mathcal{A}.$$

Theorem 2. *Let $(R(t))_{t\geq 0} \subseteq L(E)$ be a strongly continuous operator family such that there exists $\omega \geq max(0, \mathrm{abs}(|k|))$ satisfying that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is bounded. Suppose that $\mathcal{A}$ is a closed* MLO *in E and $C\mathcal{A} \subseteq \mathcal{A}C$.*

(i) Assume that $\mathcal{A}$ is a subgenerator of the global (α,ζ,k_0,C_1)-resolvent family $(R(t))_{t\geq 0}$ satisfying (3) for all $x = y \in E$. Then, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\widetilde{k_0}(\lambda) \neq 0$, the operator $I - \zeta^{-\alpha}(\lambda - ((\zeta-1)/\zeta))^{-\alpha}\mathcal{A}$ is injective, $\mathrm{R}(C) \subseteq \mathrm{R}(I - \zeta^{-\alpha}(\lambda - ((\zeta-1)/\zeta))^{-\alpha}\mathcal{A})$, *as well as*

$$\zeta^{\alpha}\widetilde{k_0}(\lambda)\left(\lambda - \frac{\zeta-1}{\zeta}\right)^{\alpha-1}\left[\zeta^{\alpha}\left(\lambda - \frac{\zeta-1}{\zeta}\right)^{\alpha} - \mathcal{A}\right]^{-1}Cx$$
$$= \int_0^{\infty} e^{-\lambda t}R(t)x\,dt, \ x \in E, \ \Re\lambda > \omega_0, \ \widetilde{k_0}(\lambda) \neq 0, \quad (12)$$

$$\left\{ \zeta^{\alpha} \left(\lambda - \frac{\zeta - 1}{\zeta} \right)^{\alpha} : \Re\lambda > \omega,\ \widetilde{k_0}(\lambda) \neq 0 \right\} \subseteq \rho_C(\mathcal{A}) \tag{13}$$

and $R(s)R(t) = R(t)R(s)$, $t,\ s \geq 0$.

(ii) *Assume* (12) *and* (13)*. Then, $\mathcal{A}$ is a subgenerator of the global $(\alpha, \zeta, k_0, C_1)$-resolvent family $(R(t))_{t \geq 0}$ satisfying* (3) *for all $x = y \in E$ and $R(s)R(t) = R(t)R(s)$, $t,\ s \geq 0$.*

Keeping in mind the last two statements, we can simply clarify the complex characterization theorem and the real characterization theorem for the generation of solution operator families for the problems $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ and $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$; see also ([36], Theorem 3.2.10, Theorem 3.2.12). Differential and analytical properties of solution operator families for the problems $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ and $(\mathrm{DFP})_{\mathbf{L}}^{\zeta}$ can be clarified following the corresponding analysis from ([36], SubSection 3.2.1); for the sequel, we need the following notion:

Definition 7.

(i) *Suppose that $\mathcal{A}$ is an MLO in X. Let $\theta \in (0, \pi]$ and let $(R(t))_{t \geq 0}$ be an (α, ζ, k_0, C)-resolvent family $(R(t))_{t \in [0,\tau)} \subseteq L(E)$ subgenerated by $\mathcal{A}$. Then, it is said that $(R(t))_{t \geq 0}$ is an analytic (α, ζ, k_0, C)-resolvent family of angle θ, if and only if there exists a function $\mathbf{R} : \Sigma_{\theta} \to L(E)$, which satisfies that, for every $x \in E$, the mapping $z \mapsto \mathbf{R}(z)x$, $z \in \Sigma_{\theta}$ is analytic as well as that:*

  (a) *$\mathbf{R}(t) = R(t)$, $t > 0$ and*

  (b) *$\lim_{z \to 0, z \in \Sigma_{\gamma}} \mathbf{R}(z)x = R(0)x$ for all $\gamma \in (0, \theta)$ and $x \in E$.*

(ii) *Let $(R(t))_{t \geq 0}$ be an analytic (α, ζ, k_0, C)-resolvent family of angle $\theta \in (0, \pi]$. Then, it is said that $(R(t))_{t \geq 0}$ is an exponentially bounded, analytic (α, ζ, k_0, C)-resolvent family of angle θ, resp. bounded analytic (α, ζ, k_0, C)-resolvent family of angle θ, if and only if for every $\gamma \in (0, \theta)$, there exists $\omega_{\gamma} \geq 0$, resp. $\omega_{\gamma} = 0$, such that the family $\{e^{-\omega_{\gamma} \Re z} \mathbf{R}(z) : z \in \Sigma_{\gamma}\} \subseteq L(E)$ is bounded. We will identify $R(\cdot)$ and $\mathbf{R}(\cdot)$ henceforth.*

Basically, the following result is the first original result of this section; it can be simply formulated for the class of analytic (α, ζ, k_0, C)-resolvent families, as well:

Theorem 3. *Suppose that the kernels $a(t)$ and $k(t)$ are given by* (11)*, $(R_1(t))_{t \in [0,\tau)} \subseteq L(X, E)$ $[(R_2(t))_{t \in [0,\tau)} \subseteq L(E)$; $(R(t))_{t \in [0,\tau)} \subseteq L(E)]$ and $k_1(t) := e^{(1-\zeta)t/\zeta} k(t)$, $t \geq 0$. Then, $(R_1(t))_{t \in [0,\tau)}$ $[(R_2(t))_{t \in [0,\tau)}$; $(R(t))_{t \in [0,\tau)}]$ is a mild (g_{α}, k_1)-regularized C_1-existence family [a mild (g_{α}, k_1)-regularized C_2-uniqueness family; a (g_{α}, k_1)-regularized C-resolvent family] subgenerated by $\zeta^{-\alpha}\mathcal{A}$ if and only if $(e^{(\zeta-1)t/\zeta} R_1(t))_{t \in [0,\tau)}$ $[(e^{(\zeta-1)t/\zeta} R_2(t))_{t \in [0,\tau)}$; $(e^{(\zeta-1)t/\zeta} R(t))_{t \in [0,\tau)}]$ is a mild $(\alpha, \zeta, k_0, C_1)$-existence family [a mild $(\alpha, \zeta, k_0, C_2)$-uniqueness family $(R_2(t))_{t \in [0,\tau)}$; an (α, ζ, k_0, C)-resolvent family $(R(t))_{t \in [0,\tau)}]$ subgenerated by $\mathcal{A}$.*

Proof. The proof is simple, and we present it only for the mild $(\alpha, \zeta, k_0, C_1)$-existence families. We know that, for every $y \in X$ and $t \in [0, \tau)$, we have

$$\left(\int_0^t g_{\alpha}(t-s) R_1(s)y, R_1(t)y - e^{\frac{1-\zeta}{\zeta}t} k(t) C_1 y \right) \in \zeta^{-\alpha}\mathcal{A}.$$

However, this is equivalent to saying that, for every $y \in X$ and $t \in [0, \tau)$, we have

$$\left(\int_0^t e^{\frac{\zeta-1}{\zeta}t} g_{\alpha}(t-s) R_1(s)y, e^{\frac{\zeta-1}{\zeta}t} R_1(t)y - k(t) C_1 y \right) \in \zeta^{-\alpha}\mathcal{A},$$

i.e., that, for every $y \in X$ and $t \in [0, \tau)$, we have

$$\left(\int_0^t \zeta^{-\alpha} e^{\frac{\zeta-1}{\zeta}(t-s)} g_\alpha(t-s) e^{\frac{\zeta-1}{\zeta}s} R_1(s)y, e^{\frac{\zeta-1}{\zeta}t} R_1(t)y - k(t)C_1 y \right) \in \mathcal{A}.$$

This simply implies the required statement. □

Remark 4. *Keeping in mind Theorem 3, we can also consider some applications of degenerate (a, k)-regularized C-resolvent families from ([36], Section 2.2, SubSection 2.3.3) to the abstract fractional Cauchy inclusions with the proportional fractional Caputo derivatives. We will skip all details concerning this issue here.*

Now, we will state and prove the following analogue of ([36], Proposition 3.2.15(i)):

Proposition 2. *Suppose that the kernels $a(t)$ and $k(t)$ are given through (11), a closed* MLO *$\mathcal{A}$ is a subgenerator of an (α, ζ, k_0, C)-regularized resolvent family $(R(t))_{t \in [0,\tau)}$, $C^{-1}f \in C([0, \tau) : E)$, $u_0 \in E$,*

$$b(t) := \zeta^\alpha \left(e^{\frac{\zeta-1}{\zeta}\cdot} g_{1-\alpha} * k_0 \right)(t), \quad t \in (0, \tau), \tag{14}$$

and

$$u(t) := R(t)u_0 + \int_0^t R(t-s)C^{-1}f(s)\, ds, \quad t \in [0, \tau). \tag{15}$$

Then, $u(t)$ is a unique solution of the abstract Cauchy inclusion (2) with $\mathcal{B} \equiv I$ and $\mathcal{F}(t) = k(t)Cu_0 + (k * f)(t)$, $t \in [0, \tau)$. If, moreover, $u_0 \in D(\mathcal{A})$ and there exists a function $f_{\mathcal{A}} \in C([0, \tau) : E)$ such that $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$, $t \in [0, \tau)$, then $u(t)$ is a unique strong solution of (2) with $\mathcal{B} \equiv I$ and $\mathcal{F}(t) = k(t)Cu_0 + (k * f)(t)$, $t \in [0, \tau)$.

Proof. The uniqueness of solutions follows from ([36], Proposition 3.2.8(ii)). To prove the existence of solutions, observe first that $u \in C([0, \tau) : E)$ and $a * b = k$. Due to our standing assumptions, we have $R(t)u_0 - k(t)Cu_0 \in \mathcal{A}(a * R)(t)u_0$, $t \in [0, \tau)$. Moreover, Lemma 1 implies that

$$(R * C^{-1}f)(t) - (k * f)(t) \in \mathcal{A}(a * R * C^{-1}f)(t), \quad t \in [0, \tau).$$

This simply implies the first statement with $u_{\mathcal{B}} = u$ and $u_{a,\mathcal{A}} = u - \mathcal{F}$; cf. Definition 1(ii). Suppose now that $u_0 \in D(\mathcal{A})$ and there exists a function $f_{\mathcal{A}} \in C([0, \tau) : E)$ with the prescribed properties. Then, there exists $v_0 \in E$ such that $(u_0, v_0) \in \mathcal{A}$ and therefore $(R(t)u_0, R(t)v_0) \in \mathcal{A}$ for all $t \in [0, \tau)$. Then, $u(t)$ is a strong solution of (2) with $\mathcal{B} \equiv I$ and $\mathcal{F}(t) = k(t)Cu_0 + (k * f)(t)$, $t \in [0, \tau)$ since we can take the function

$$u_{\mathcal{A}}(t) = R(t)v_0 + \int_0^t R(t-s)f_{\mathcal{A}}(s)\, ds, \quad t \in [0, \tau)$$

in the third part of Definition 1 (we need to apply Lemma 1 once more here). □

Remark 5. *We feel it is our duty to say that we made a small mistake by stating that the solutions of fractional Cauchy problems constructed in ([41], Proposition 2.1.32 and [36], Proposition 3.2.15) are continuously differentiable on the interval $(0, \tau)$, which is not true in general. Consider, for example, the situation of ([41], Proposition 2.1.32(i)). Let $E := l^1$, the Banach space of norm-summable numerical sequences (x_k) equipped with the norm $\|(x_k)\| := \sum_{k=1}^{\infty} |x_k|$, let $0 < \alpha < 1$, and let a closed linear operator A_α on E be defined through $D(A_\alpha) := \{(x_k) \in E : \sum_{k=1}^{\infty} k|x_k| < +\infty\}$ and*

$A_\alpha(x_k) := (e^{i\alpha\pi/2}kx_k)$, $(x_k) \in D(A_\alpha)$. Then, E. Bazhlekova has proved, in ([38], Example 2.24), that the operator A_α generates a bounded (g_α, I)-regularized resolvent family $(S_\alpha(t))_{t\geq 0}$, given by

$$S_\alpha(t)(x_k) := \Big(E_\alpha\big(e^{i\alpha\pi/2}kt^\alpha\big)x_k\Big), \quad t \geq 0, \ (x_k) \in E,$$

as well as that the operator $A_\alpha + I$ does not generate an exponentially bounded (g_α, I)-regularized resolvent family; here, $E_\alpha(z)$ denotes the Mittag–Leffler function (see [38,41] for the notion used below). Suppose now that $(x_k) \in D(A_\alpha)$ and $\sum_{k=1}^{\infty} k^{(1-\alpha)/\alpha}|x_k| = +\infty$. If the mapping $t \mapsto S_\alpha(t)(x_k)$, $t > 0$ is differentiable at some point $t > 0$, then we must have

$$\frac{d}{dt}S_\alpha(t)(x_k) = \Big(t^{\alpha-1}E_{\alpha,\alpha}\big(e^{i\alpha\pi/2}kt^\alpha\big)x_k\Big).$$

Due to the asymptotic expansion formula for the Mittag–Leffler functions (see, e.g., ([41], Theorem 1.3.1, (17)-(19)), we have

$$\left|t^{\alpha-1}E_{\alpha,\alpha}\big(e^{i\alpha\pi/2}kt^\alpha\big)\right| \sim \frac{1}{\alpha}t^{\alpha-1}\left|\big(e^{i\alpha\pi/2}kt^\alpha\big)^{(1-\alpha)/\alpha}\right| \sim \frac{1}{\alpha}k^{(1-\alpha)/\alpha},$$

as $k \to +\infty$. This would imply $\sum_{k=1}^{\infty} k^{(1-\alpha)/\alpha}|x_k| < +\infty$, which is a contradiction. Therefore, the mapping $t \mapsto S_\alpha(t)(x_k)$, $t > 0$ is not differentiable at any point $t > 0$.

The interested reader may try to formulate some analogues of ([36], Proposition 3.2.15(ii)) in our new framework; we continue by stating the following result:

Theorem 4. *Suppose that the requirements of* Proposition 2 *hold with $k_0(t) = \delta(t)$ being the Dirac delta distribution, $u_0 \in D(\mathcal{A})$ and the existence of a function $f_{\mathcal{A}} \in C([0,\tau) : E)$ such that $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$, $t \in [0,\tau)$. Then, the function $u(t)$, given by* (15)*, is a solution of the abstract fractional Cauchy problem* $(\text{DFP})^{\zeta}_{\mathbf{R}}$ *with $B = I$ and $\mathcal{F}(t) = (b * f)(t)$, $t \in [0,\tau)$, where the function $b(t)$ is given by* (14)*.*

Proof. By Definition 5(i), it suffices to prove that the fractional derivative $\mathbf{D}_t^{\alpha,\zeta}u(t)$ is well defined as well as that

$$\mathbf{D}_t^{\alpha,\zeta}u(t) = R(t)v_0 + \big(R * f_{\mathcal{A}}\big)(t) + (b * f)(t), \quad t \in [0,\tau). \tag{16}$$

Keeping in mind our consideration from Remark 2, we have that $\mathbf{D}_t^{\alpha,\zeta}u(t)$ is well defined if and only if

$$g_{1-\alpha} * \Big[e^{\frac{1-\zeta}{\zeta}\cdot}\Big(R(\cdot)u_0 + \big(R * C^{-1}f\big)(\cdot)\Big) - Cu_0\Big](\cdot) \in C^1([0,\tau) : E).$$

Due to Theorem 3, we have that $(e^{\frac{1-\zeta}{\zeta}t}R(t))_{t\in[0,\tau)}$ is a (g_α, k_1)-regularized C-resolvent family with a subgenerator $\zeta^{-\alpha}\mathcal{A}$. Since $k_1(0) = k(0) = 1$, we have $e^{\frac{1-\zeta}{\zeta}t}R(t)u_0 - Cu_0 = \zeta^{-\alpha}\int_0^t g_\alpha(t-s)e^{\frac{1-\zeta}{\zeta}s}R(s)v_0\,ds$, where $v_0 \in \mathcal{A}u_0$ is chosen arbitrarily. This simply implies that

$$g_{1-\alpha} * e^{\frac{1-\zeta}{\zeta}\cdot}\Big[R(\cdot)u_0 - Cu_0\Big](\cdot) \in C^1([0,\tau) : E);$$

therefore, we need to prove that

$$g_{1-\alpha} * e^{\frac{1-\zeta}{\zeta}\cdot}\Big(R * C^{-1}f\Big)(\cdot) \in C^1([0,\tau) : E).$$

We have

$$
\begin{aligned}
&\Big[g_{1-\alpha} * e^{\frac{1-\zeta}{\zeta}\cdot}\Big(R * C^{-1}f\Big)(\cdot)\Big](t) \\
&= \int_0^t g_{1-\alpha}(t-s)e^{\frac{1-\zeta}{\zeta}s}\int_0^s\Bigg[k(s-r)f(r) \\
&\quad + \zeta^{-\alpha}\int_0^{s-r} g_\alpha(s-r-v)e^{\frac{\zeta-1}{\zeta}(s-r-v)}R(v)f_{\mathcal{A}}(r)\,dv\Bigg]\,dr\,ds \\
&= \int_0^t g_{1-\alpha}(t-s)\int_0^s e^{\frac{1-\zeta}{\zeta}r}f(r)\,dr\,ds \\
&\quad + \zeta^{-\alpha}\left(g_{1-\alpha} * g_\alpha * \Big[e^{\frac{1-\zeta}{\zeta}\cdot}R(\cdot)\Big] * \Big[e^{\frac{1-\zeta}{\zeta}\cdot}f_{\mathcal{A}}\Big]\right)(t) \\
&= \int_0^t g_{2-\alpha}(t-s)e^{\frac{1-\zeta}{\zeta}s}f(s)\,ds \\
&\quad + \zeta^{-\alpha}\left(g_1 * \Big[e^{\frac{1-\zeta}{\zeta}\cdot}R(\cdot)\Big] * \Big[e^{\frac{1-\zeta}{\zeta}\cdot}f_{\mathcal{A}}\Big]\right)(t), \quad t \in [0,\tau),
\end{aligned}
$$

where we applied the partial integration for the first addend in the last equality. In view of the obtained formula, we have:

$$
\begin{aligned}
&\frac{d}{dt}\Big[g_{1-\alpha} * e^{\frac{1-\zeta}{\zeta}\cdot}\Big(R * C^{-1}f\Big)(\cdot)\Big](t) = \int_0^t g_{1-\alpha}(t-s)e^{\frac{1-\zeta}{\zeta}s}f(s)\,ds \\
&+\zeta^{-\alpha}\left(\Big[e^{\frac{1-\zeta}{\zeta}\cdot}R(\cdot)\Big] * \Big[e^{\frac{1-\zeta}{\zeta}\cdot}f_{\mathcal{A}}\Big]\right)(t), \quad t \in [0,\tau),
\end{aligned}
$$

which implies that the fractional derivative $\mathbf{D}_t^{\alpha,\zeta}u(t)$ is well defined. Since the both sides of Equation (16) are well defined, its equality is equivalent with the corresponding equality obtained by convoluting the both sides of (16) with $a(t)$. This is equivalent to saying that $u(t)$ is a strong solution of the associated Volterra inclusion (2), as easily approved, so that an application of Proposition 2 completes the proof. $\square$

As a consequence of the last theorem, we have that the solution of the abstract fractional inclusion $(\mathrm{DFP})_{\mathbf{R}}^{\zeta}$ with $B = I$ can be expected, in the usually considered situation, only if the function $\mathcal{F}(t)$ belongs to the range of convolution transform $b * \cdot$, where the function $b(t)$ is given by (14). It is clear that Theorem 4 gives us rise to define, under certain logical assumptions, the mild solution of the abstract fractional Cauchy inclusion ($\alpha \in (0,1)$, $\zeta \in (0,1]$)

$$
\begin{cases}
(\mathrm{DFP})_{\mathbf{R},s}^{\zeta} : \mathbf{D}_t^{\alpha,\zeta}u(t) \in \mathcal{A}u(t) + \int_0^t b(t-s)f(s,u(s))\,ds, \quad t \in [0,\tau), \\
u(0) = Cu_0,
\end{cases}
$$

as any continuous function $u : [0,\tau) \to E$ such that

$$
u(t) = R(t)u_0 + \int_0^t R(t-s)C^{-1}f(s,u(s))\,ds, \quad t \in [0,\tau).
$$

We will analyze the qualitative properties of solutions of the abstract semilinear Cauchy inclusion $(\mathrm{DFP})_{\mathbf{R},s}^{\zeta}$ elsewhere.

3.2. Some Applications to the Abstract Volterra Integro-Differential Inclusions

It is clear that Theorem 3, Proposition 2 and Theorem 4 can be applied to the various classes of the abstract Volterra integro-differential inclusions and the abstract fractional Cauchy inclusions with the proportional Caputo fractional derivatives:

1. (cf. [36], Example 3.2.23) Suppose that $0 < \alpha < 1$, the closed linear operators A and B satisfy the condition ([40], (3.14)) with $\alpha = 1$ and some real constants $0 < \beta \leq 1$, $\gamma \in \mathbb{R}$ and c, $C > 0$ (in our notation, we have $A = L$ and $B = M$). Then, we know that the multivalued linear operator AB^{-1} generates an exponentially bounded, analytic $(g_1, g_{1+\sigma})$-regularized I-resolvent family of angle $\Sigma_{\text{arctan}(1/c)}$, provided that $\sigma > 1 - \beta$.

The subordination principle then implies (see, e.g., ([44], Theorem 3.9) and [36]) that AB^{-1} generates an exponentially bounded, analytic $(g_\alpha, g_{1+\alpha\sigma})$-regularized I-resolvent family of angle $\theta \geq \min((\pi/2), (\pi/2)(\alpha^{-1} - 1))$ for any $\sigma > 1 - \beta$; the value of this angle can be probably increased using the argumentation contained in the proof of ([44], Theorem 3.10); however, we will not discuss this question here. For example, we can consider the well-posedness of the abstract Volterra integral inclusions associated with the following Poisson heat equation in the space $E = L^p(\Omega)$:

$$(P)_b : \begin{cases} \frac{\partial}{\partial t}[m(x)v(t,x)] = \Delta v - bv, \quad t \geq 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$ and $1 < p < \infty$. Let B be the multiplication in $L^p(\Omega)$ with $m(x)$ and let $A = \Delta - b$ act with the Dirichlet boundary conditions; these operators satisfy the general requirements of this example described above. Set $\mathcal{A} := \zeta^\alpha AB^{-1}$.

Then, Theorem 3 implies that the multivalued linear operator $\mathcal{A}$ generates an exponentially bounded, analytic (α, ζ, k_0, I)-resolvent family of angle $\theta \geq \min((\pi/2), (\pi/2)(\alpha^{-1} - 1))$ for any $\sigma > 1 - \beta$; here $k_0(t) = e^{\frac{\zeta-1}{\zeta}t} g_{\alpha\sigma}(t)$, $t > 0$, $b(t) = \zeta^\alpha e^{\frac{\zeta-1}{\zeta}t} g_{1-\alpha+\alpha\sigma}(t)$, $t > 0$, and $k(t) = e^{\frac{\zeta-1}{\zeta}t} g_{1+\alpha\sigma}(t)$, $t > 0$. Then, Proposition 2 is applicable for any $u_0 \in E$ and $f \in C([0,\tau) : E)$. We can similarly apply Proposition 2 to the certain classes of the abstract Volterra integral inclusions with the almost sectorial operators (cf. [36,41] and references cited therein).

2. (cf. [41], Section 2.5) In this part, we briefly explain how we can apply our theoretical results in the analysis of the abstract fractional Cauchy problems with the proportional Caputo fractional derivatives and the abstract differential operators generating fractional resolvent families (cf. also [36], SubSection 3.10.1). Suppose that the requirements of ([41], Theorem 2.5.3) are satisfied with $\alpha \in (0,1)$. Then, there exists an injective operator $C \in L(E)$ such that the single-valued linear operator $\overline{P(A)}$ generates an exponentially bounded (g_α, C)-regularized resolvent family. In our concrete situation, we have that $k_0(t) = \delta(t)$ is the Dirac delta distribution and $b(t) = \zeta^\alpha e^{\frac{\zeta-1}{\zeta}t} g_{1-\alpha}(t)$, $t > 0$.

Due to Theorem 3, the operator $\zeta^\alpha \overline{P(A)}$ generates an exponentially bounded $(\alpha, \zeta, \delta, C)$-regularized resolvent family. If $u_0 \in D(\overline{P(A)})$ and the function $t \mapsto C^{-1}f(t)$, $t \in [0,\tau)$ is continuous, then Theorem 4 implies that there exists a unique solution of the abstract fractional Cauchy problem $(\text{DFP})^\zeta_{\mathbf{R}}$ with $B = I$, $\mathcal{A} = \zeta^\alpha \overline{P(A)}$ and $\mathcal{F}(t) = (b * f)(t)$, $t \in [0,\tau)$.

As the many research studies performed thus far, it is clear how we can use this result in the analysis of the abstract fractional Cauchy problems in L^p-spaces with the proportional Caputo fractional derivatives and the abstract differential (non-elliptic, in general) differential operators in L^p-spaces with the constant coefficients; for example, we can consider the well-posedness of the following abstract fractional Cauchy problem:

$$\begin{cases} \mathbf{D}_t^{\alpha,\zeta} u(t,x) = \zeta^\alpha e^{\frac{i(2-\alpha)\pi}{2}} \Delta u(t,x) + \zeta^\alpha \left(e^{\frac{\zeta-1}{\zeta} \cdot} * f \right)(t),\ \alpha \in (0,1),\ \zeta \in (0,1],\ t \geq 0, \\ u(0) = \left(I - \Delta \right)^{-\gamma} u_0, \end{cases}$$

provided that $1 < p < \infty$ and $\gamma \geq n|(1/p) - (1/2)|/\alpha$ (see also [41], Example 2.5.6).

3. Concerning the classes of mild $(\alpha, \zeta, k_0, C_1)$-existence families and the mild $(\alpha, \zeta, k_0, C_2)$-uniqueness families, we will only emphasize that Theorems 3 and 4 can be successfully applied in the analysis of the fractional heat equation with the proportional Caputo fractional derivative in the space $E := \{f \in C(\mathbb{R}) : \lim_{|x| \to \infty} e^{x^2} f(x) = 0\}$; see the final part of ([41], Section 2.8) for more details.

In our new framework, some other applications can be given using ([41], Theorem 2.3.1(ii), Theorem 2.3.3, Remark 2.5.4(ii), Example 2.6.39).

4. Suppose that $k_0(t) = \delta(t)$, the Dirac Delta distribution and consider the situation of Theorem 3; then, $k_1(t) \equiv 1$. If the corresponding $(g_\alpha, 1)$-regularized C-resolvent family $(R_1(t))_{t \geq 0}$ subgenerated by $\zeta^{-\alpha} \mathcal{A}$ exists and satisfies that $\|R_1(t)\| \leq M e^{\omega t}$, $t \geq 0$ for some real numbers $M > 0$ and $\omega \in [0, (1-\zeta)/\zeta)$, then the corresponding $(\alpha, \zeta, \delta, C)$-regularized resolvent family $(R(t))_{t \geq 0}$ subgenerated by $\mathcal{A}$ is exponentially decaying. Therefore, Proposition 2 and Theorem 4 can be successfully applied in the analysis of the existence and uniqueness of asymptotically almost periodic (automorphic) type solutions of the corresponding abstract Volterra integral inclusions and the abstract fractional Cauchy inclusions with the proportional Caputo fractional derivatives; see., e.g., [45], Lemma 2.13, [46], Lemma 4.1 and [20], Propositions 2.6.13, 2.7.5, 2.11.10, 3.5.4.

4. Almost Periodic Type Solutions of Semilinear Proportional Caputo Fractional Differential Equations

Let $\alpha \in (0,1)$ and $\zeta \in (0,1]$. In this section, we continue the investigation of R. Agarwal, S. Hristova and D. O'Regan [12] by studying the almost periodic-type solutions of the following proportional Caputo fractional differential equation with $\mathcal{A} \equiv 0$:

$$\begin{cases} {}_0^c D^{\alpha,\zeta} u(t) = f(t, u(t)), \quad \alpha \in (0,1),\ \zeta \in (0,1],\ t \geq 0, \\ u(0) = u_0. \end{cases} \tag{17}$$

By a mild solution of (17), we mean any continuous function $u : [0, \infty) \to X$ such that (see also [12], Lemma 3):

$$u(t) = u_0 e^{\frac{\zeta-1}{\zeta} t} + \frac{1}{\zeta^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1} f(s, u(s))\, ds, \quad t \geq 0. \tag{18}$$

We will occasionally use the following conditions:

(C1) The function $f : I \times X \to X$ is continuous,

$$\int_0^\omega f(\omega - s, x) e^{\frac{\zeta-1}{\zeta} s} s^{\alpha-1}\, ds = 0, \quad x \in X, \tag{19}$$

and, for every bounded subset B of X, we have $\sup_{t \geq 0; x \in B} \|f(t,x)\| < +\infty$.

(C2) There exists a finite real constant $L_f > 0$ such that

$$\|f(t,u) - f(t,v)\| \leq L_f \|u - v\| \quad \text{for all } t \geq 0 \text{ and } u,\ v \in X.$$

(C3) $c = e^{\frac{\zeta-1}{\zeta}\omega}$.

Then, we have the following (by the proof of the Banach contraction principle, the unique solution of (17) may be approximated by means of the iterative sequence given by the operator T defined below):

Proposition 3. *Let* (C1)–(C3) *hold and let* $L_f < (1-\zeta)^\alpha$. *Then, there exists a unique solution* $u \in \Phi_{\omega,c}$ *of* (17).

Proof. The solution of (17) is given by (18). We define the operator $T : \Phi_{\omega,c} \to \Phi_{\omega,c}$ by

$$(Tu)(t) := u_0 e^{\frac{\zeta-1}{\zeta}t} + \frac{1}{\zeta^\alpha \Gamma(\alpha)} \int\limits_0^t e^{\frac{\zeta-1}{\zeta}(t-s)}(t-s)^{\alpha-1} f(s,u(s))\,ds, \quad t \geq 0.$$

Let us show that T is well defined. Thus, let $u \in \Phi_{\omega,c}$. Then, a straightforward computation involving conditions (19) and (C3) shows that $(Tu)(\omega) = c(Tu)(0)$. We have that $Tu(\cdot)$ is a bounded function since we assumed that, for every bounded subset B of X, we have $\sup_{t\geq 0; x\in B} \|f(t,x)\| < +\infty$ and

$$\big[\Gamma(\alpha)\big]^{-1} \int_0^\infty e^{\frac{\zeta-1}{\zeta}(t-s)}(t-s)^{\alpha-1}\,ds = \big[(1-\zeta)/\zeta\big]^{-\alpha} < +\infty;$$

moreover, it is elementary to prove that $Tu(\cdot)$ is a continuous function. Therefore, T is well defined; since we assumed that $L_f < (1-\zeta)^\alpha$, it can be simply shown that the mapping T is a contraction, which implies that Equation (17) has a unique solution $u \in \Phi_{\omega,c}$ due to the Banach contraction principle. □

We continue by providing the following illustrative example:

Example 2. *Suppose that* $X = \mathbb{C}$ *and* $f(t,x) = f_1(t)f_2(x)$, *where* $f_1 \in C_b([0,\infty) : X)$ *and* $f_2(\cdot)$ *is Lipschitz continuous on* $\mathbb{R}$ *with the Lipschitz constant* $L_{f_2} > 0$ *satisfying* $\|f_1\|_\infty L_{f_2} < (1-\zeta)^\alpha$. *If*

$$\int_0^\omega f_1(\omega-s)e^{\frac{\zeta-1}{\zeta}s}s^{\alpha-1}\,ds = 0,$$

then Proposition 3 can be simply applied.

It is clear that the proportional fractional integrals, the proportional Caputo fractional derivatives and the abstract Cauchy fractional problem (17) can be considered for the functions defined on the finite interval $[0,\omega]$. Define $\Psi_{\omega,c} := \{u \in C[0,\omega] : u(\omega) = cu(0)\}$; then $\Psi_{\omega,c}$ is a Banach space when equipped with the sup-norm.

For the sequel, we need the following well-known result:

Theorem 5 (Schauder fixed-point theorem [47])**.** *Let X be a Banach space and $\Omega \subseteq X$ be a convex, closed and bounded set. If $T : \Omega \to \Omega$ is a continuous operator such that $T(\Omega)$ is pre-compact, then T has at least one fixed point in Ω.*

Consider the following conditions:

(C1)′ The function $f : [0,\omega] \times X \to X$ is continuous and

$$\int_0^\omega f(\omega-s,x)e^{\frac{\zeta-1}{\zeta}s}s^{\alpha-1}\,ds = 0, \quad x \in X.$$

(C4) There exist real constants $C_1,\ C_2 > 0$ such that

$$\|f(t,u)\| \leq C_1\|u\| + C_2 \quad \text{for all } t \in [0,\omega] \text{ and } u \in X.$$

In the subsequent result, we apply the Schauder fixed-point theorem to prove the existence of a solution $u \in \Psi_{\omega,c}$ of the problem (17) on $[0,\omega]$:

Theorem 6. *Let the conditions* (C1)′ *and* (C3)–(C4) *hold and let*

$$I_\omega := \int_0^\omega e^{\frac{\zeta-1}{\zeta}s} s^{\alpha-1}\, ds.$$

If

$$C_1 I_\omega < \zeta^\alpha \Gamma(\alpha),$$

then the proportional Caputo fractional differential Equation (17) *has at least one solution* $u \in \Psi_{\omega,c}$ *on* $[0,\omega]$.

Proof. Define $B_m := \{u \in \Psi_{\omega,c} \ : \ \|u\| \leq m\}$, where

$$m\left(1 - \frac{C_1 I_\omega}{\zeta^\alpha \Gamma(\alpha)}\right) > \|u_0\| + \frac{C_2 I_\omega}{\zeta^\alpha \Gamma(\alpha)}.$$

Let T be the operator defined as in the proof of Proposition 3. For every $t \in [0,\omega]$ and $u \in B_m$, we have

$$\begin{aligned}
\|(Tu)(t)\| &\leq \|u_0\| + \frac{1}{\zeta^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1} \|f(s,u(s))\|\, ds \\
&\leq \|u_0\| + \frac{1}{\zeta^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1} \big(C_1\|u(s)\| + C_2\big)\, ds \\
&\leq \|u_0\| + \frac{1}{\zeta^\alpha \Gamma(\alpha)} (C_1\|u\| + C_2) I_\omega < m.
\end{aligned}$$

Thus, $\|Tu\| \leq m$ and $T(B_m) \subseteq B_m$.

Now, we will prove that the operator T is continuous on B_m. Let (u_n) be a sequence in B_m such that $u_n \to u$ on B_m, when $n \to \infty$. Since $f(t,x)$ is a continuous function, we have $f(s,u_n(s)) \to f(s,u(s))$, when $n \to \infty$. Hence,

$$e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1} f(s,u_n(s)) \to e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1} f(s,u(s)), \quad \text{as} \quad n \to \infty.$$

By (C4), we have

$$\begin{aligned}
&\int_0^t \left\| e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1} f(s,u_n(s)) - e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1} f(s,u(s)) \right\| ds \\
&\leq 2(C_1 m + C_2) \int_0^t e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1}\, ds \leq 2(C_1 m + C_2) I_\omega < +\infty.
\end{aligned}$$

Now, by the Lebesgue dominated convergence theorem, we have

$$\int_0^t \left\| e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1} f(s,u_n(s)) - e^{\frac{\zeta-1}{\zeta}(t-s)} (t-s)^{\alpha-1} f(s,u(s)) \right\| ds \to 0,$$

as $n \to \infty$. This simply implies that the operator T is continuous on B_m.

Next, we prove that the operator T is pre-compact. For any $0 \le s_1 \le s_2$ and $u \in B_m$, we have

$$\begin{aligned}
&\big\|(Tu)(s_1) - (Tu)(s_2)\big\| \le \|u_0\| \cdot \Big|e^{\frac{\zeta-1}{\zeta}s_1} - e^{\frac{\zeta-1}{\zeta}s_2}\Big| \\
&+ \Bigg\| \frac{1}{\zeta^\alpha \Gamma(\alpha)} \int_0^{s_1} e^{\frac{\zeta-1}{\zeta}(s_1-s)} (s_1-s)^{\alpha-1} f(s,u(s))\, ds \\
&- \frac{1}{\zeta^\alpha \Gamma(\alpha)} \int_0^{s_2} e^{\frac{\zeta-1}{\zeta}(s_2-s)} (s_2-s)^{\alpha-1} f(s,u(s))\, ds \Bigg\| \\
&\le \|u_0\| \cdot \Big|e^{\frac{\zeta-1}{\zeta}s_1} - e^{\frac{\zeta-1}{\zeta}s_2}\Big| \\
&+ \frac{1}{\zeta^\alpha \Gamma(\alpha)} \int_0^{s_1} \Big(e^{\frac{\zeta-1}{\zeta}(s_1-s)} (s_1-s)^{\alpha-1} - e^{\frac{\zeta-1}{\zeta}(s_2-s)} (s_2-s)^{\alpha-1} \Big) \|f(s,u(s))\|\, ds \\
&+ \frac{1}{\zeta^\alpha \Gamma(\alpha)} \int_{s_1}^{s_2} e^{\frac{\zeta-1}{\zeta}(s_2-s)} (s_2-s)^{\alpha-1} \|f(s,u(s))\|\, ds \\
&\le \|u_0\| \cdot \Big|e^{\frac{\zeta-1}{\zeta}s_1} - e^{\frac{\zeta-1}{\zeta}s_2}\Big| \\
&+ \frac{C_1 m + C_2}{\zeta^\alpha \Gamma(\alpha)} \int_0^{s_1} \Big(e^{\frac{\zeta-1}{\zeta}(s_1-s)} (s_1-s)^{\alpha-1} - e^{\frac{\zeta-1}{\zeta}(s_2-s)} (s_2-s)^{\alpha-1} \Big)\, ds \\
&+ \frac{C_1 m + C_2}{\zeta^\alpha \Gamma(\alpha)} \int_{s_1}^{s_2} e^{\frac{\zeta-1}{\zeta}(s_2-s)} (s_2-s)^{\alpha-1}\, ds \\
&\le \|u_0\| \cdot \Big|e^{\frac{\zeta-1}{\zeta}s_1} - e^{\frac{\zeta-1}{\zeta}s_2}\Big| \\
&+ \frac{C_1 m + C_2}{\zeta^\alpha \Gamma(\alpha)} \int_0^{s_1} \Big(e^{\frac{\zeta-1}{\zeta}(s_1-s)} (s_1-s)^{\alpha-1} - e^{\frac{\zeta-1}{\zeta}(s_2-s)} (s_2-s)^{\alpha-1} \Big)\, ds \\
&+ \frac{C_1 m + C_2}{\zeta^\alpha \Gamma(\alpha+1)} (s_2 - s_1)^\alpha \to 0,
\end{aligned}$$

as $s_1 \to s_2$, independently of $u \in B_m$; here, we can apply the dominated convergence theorem for the second addend. Therefore, $T(B_m)$ is equicontinuous. Since $T(B_m)$ is uniformly bounded, the Arzelá–Ascoli theorem (see, e.g., [48]) implies that $T(B_m)$ is pre-compact. Using Schauder's fixed-point theorem, we finally obtain that the proportional Caputo fractional differential Equation (17) has at least one solution $u \in \Psi_{\omega,c}$. □

5. Nonexistence of (ω, c)-Periodic Solutions of (17) and Nonexistence of Poisson Stable-Like Solutions of (17)

In this section, we consider the nonexistence of (ω, c)-periodic solutions of (17) and the nonexistence of Poisson-stable-like solutions of (17). Before doing this, we will continue and slightly extend the results of I. Area, J. Losada and J. J. Nieto [35,37] concerning the quasi-periodic properties of the Riemann–Liouville fractional integrals (see also I. Area, J. Losada, J. J. Nieto [49] and J. M. Jonnalagadda [50] for the discrete analogues). For simplicity, we will not thoroughly analyze the quasi-periodic properties of proportional Caputo fractional derivatives.

5.1. On Quasi-Periodic Properties of Proportional Fractional Integrals

Let us consider first the statement of Proposition 3 with $f(t,x) \equiv f(t)$, $t \geq 0$. If (C3) holds, then the existence of a unique nonzero (ω, c)-periodic solution of (17) can be expected only if

$$\int_t^{t+\omega} f(t+\omega-s)e^{\frac{\zeta-1}{\zeta}s}s^{\alpha-1}\,ds = 0, \quad t \geq 0,$$

i.e.,

$$\int_0^{\omega} f(\omega-s)e^{\frac{\zeta-1}{\zeta}(t+s)}(t+s)^{\alpha-1}\,ds = 0, \quad t \geq 0, \tag{20}$$

which is a restrictive assumption. In ([37], Theorem 1), the authors have proven that the validity of (20) with $\zeta = 1$ and $f \in L^1_{loc}([0,\infty) : \mathbb{R})$ being a nonzero ω-periodic function implies that the Riemann–Liouville integral $J_t^\alpha f(t) = ({}_0I^{\alpha,1} f)(\cdot)$ cannot be an ω-periodic function for any $\alpha \in (0,1)$; moreover, the authors have proven, in ([37], Section 4), that $({}_0I^{\alpha,1} f)(\cdot)$ cannot be ω'-periodic for any $\alpha \in (0,1)$ and $\omega' > 0$ (see also ([37], Corollary 2) for a fractional derivative analogue of the first-mentioned result). Therefore, it is logical to ask whether these results continue to hold for an arbitrary value of parameter $\zeta \in (0,1)$.

Before considering this issue, we would like to state and prove a new theoretical result about the quasi-periodic properties of the Riemann–Liouville fractional integrals of essentially bounded ω-periodic functions. Suppose that $\alpha \in (0,1)$, $\omega > 0$ and $f : [0,\infty) \to X$ is a non-zero essentially bounded ω-periodic function. Then, ([35], Lemma 3) continues to hold for $f(\cdot)$, as it can be simply verified, so that the function $J_t^\alpha f(\cdot)$ is S-asymptotically ω-periodic (cf. [51] for the notion). If we suppose that the function $J_t^\alpha f(\cdot)$ is Poisson stable, this would imply by ([51], Lemma 3.1) that the function $J_t^\alpha f(\cdot)$ is ω-periodic. This will be used in the proof of the following proper extension of ([35], Theorem 9), which has been formulated in a slightly different manner as Theorem 2.3.48 of [25]:

Theorem 7. *Suppose that $\alpha \in (0,1)$, $\omega > 0$ and $f : [0,\infty) \to X$ is a non-zero essentially bounded ω-periodic function. Then, $J_t^\alpha f(\cdot)$ cannot be Poisson stable (a restriction of an almost automorphic function to the non-negative real line).*

Proof. We will first consider the Poisson stable functions. Suppose that $J_t^\alpha f(\cdot)$ is Poisson stable and $x^* \in X^*$ is an arbitrary functional. Let $\langle x^*, f(\cdot)\rangle = a(\cdot) + ib(\cdot)$, where $a(\cdot)$ and $b(\cdot)$ are real-valued functions. Then, the function $J_t^\alpha \langle x^*, f(\cdot)\rangle = J_t^\alpha a(\cdot) + iJ_t^\alpha b(\cdot)$ is Poisson stable because $J_t^\alpha \langle x^*, f(\cdot)\rangle = \langle x^*, J_t^\alpha f(\cdot)\rangle$, which further implies that the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are Poisson stable. Since $a(\cdot)$ and $b(\cdot)$ are essentially bounded functions of period ω, the above discussion implies that $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are periodic functions of period ω. Then, we can apply ([37], Theorem 1) in order to see that $a(\cdot) \equiv b(\cdot) \equiv 0$. This implies $\langle x^*, f(\cdot)\rangle \equiv 0$ and therefore $f(\cdot) \equiv 0$ since x^* was arbitrary.

Suppose now that $J_t^\alpha f(\cdot)$ is a restriction of an almost automorphic function to the non-negative real line. For the remainder of the proof, it is essential to observe that the statements of ([35], Lemma 1, Theorem 5) hold not only for continuous periodic functions but also for essentially bounded periodic functions (let us only note here that the functions $\varphi_n(t)$ and $\Phi_n(t)$ defined at the beginning of the proof of ([35], Theorem 5) are continuous for any essentially bounded T-periodic function $f(t)$, as a simple computation shows.

Keeping in mind our assumptions, we obtain now that the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are asymptotically ω-periodic on the non-negative real line so that there exist two continuous ω-periodic functions $g_{a,b} : \mathbb{R} \to \mathbb{R}$ and two continuous functions $\psi_{a,b} : [0,\infty) \to \mathbb{R}$ vanishing at plus infinity so that $J_t^\alpha a(t) = g_a(t) + \psi_a(t)$, $t \geq 0$ and $J_t^\alpha b(t) = g_b(t) + \psi_b(t)$, $t \geq 0$. This is impossible because the almost automorphic function $J_t^\alpha a(\cdot) - g_a(\cdot)$ $[J_t^\alpha b(\cdot) - g_b(\cdot)]$ cannot vanish at plus infinity on account of the supremum formula ([20], Lemma 3.9.9). Therefore, $a(\cdot) \equiv b(\cdot) \equiv 0$, $\langle x^*, f(\cdot)\rangle \equiv 0$, and therefore $f(\cdot) \equiv 0$. $\square$

Applying the trick used in the first part of the proof and the well known fact that a weakly bounded set in a locally convex space is bounded, we may conclude that the statements of ([37], Theorem 1, Corollary 2 and [35], Lemma 2, Lemma 3; Proposition 1, Proposition 2; Theorem 2, Theorem 3, Theorem 4 and Theorem 8) hold in the vector-valued case (concerning the above-mentioned statements from [35], it seems plausible that the continuity of function $f(\cdot)$ in their formulations can be replaced with the essential boundedness). It is clear that ([35], Corollary 1) cannot be reformulated even for the complex-valued functions and, regarding the main structural results established in [35,37], it remains to be considered whether the statements of ([35], Theorem 5, Theorem 6 and Theorem 7) hold in the vector-valued case. We will analyze this question elsewhere.

Let us come back now to the question proposed at the end of the first paragraph of this subsection. First, we prove that the statements of ([37], Theorem 1) and Theorem 7 can be simply transferred to the proportional Caputo fractional integrals. More precisely, we have the following:

Theorem 8. *Let $\zeta \in (0,1)$, $\alpha \in (0,1)$ and* (C3) *hold.*

(i) *Suppose that $f : [0,\infty) \to \mathbb{R}$ is a non-zero locally integrable (ω, c)-periodic function. Then, the function $({}_0I^{\alpha,\zeta} f)(\cdot)$ cannot be (ω, c)-periodic.*

(ii) *Suppose that $f : [0,\infty) \to X$ is a non-zero essentially bounded (ω, c)-periodic function. Then, the function $e^{\frac{1-\zeta}{\zeta}\cdot}({}_0I^{\alpha,\zeta} f)(\cdot)$ cannot be Poisson stable (a restriction of an almost automorphic function to the non-negative real line).*

(iii) *Suppose that $f : [0,\infty) \to X$ is a non-zero essentially bounded (ω, c)-periodic function. Then, the function $e^{\frac{1-\zeta}{\zeta}\cdot}({}_0I^{\alpha,\zeta} f)(\cdot)$ is S-asymptotically ω-periodic.*

Proof. We will prove only (i). The function $g(t) := e^{\frac{1-\zeta}{\zeta}t} f(t)$, $t \geq 0$ is nonzero, locally integrable and ω-periodic, as easily shown. An application of ([37], Theorem 1) shows that the function $t \mapsto \int_0^t g_\alpha(t-s)e^{\frac{1-\zeta}{\zeta}s} f(s)\, ds$, $t \geq 0$ cannot be ω-periodic, i.e., the function $t \mapsto e^{\frac{\zeta-1}{\zeta}t} \int_0^t g_\alpha(t-s)e^{\frac{1-\zeta}{\zeta}s} f(s)\, ds$, $t \geq 0$ cannot be (ω, c)-periodic. This implies the required. □

Remark 6. *We cannot use (iii) in order to prove that the function $({}_0I^{\alpha,\zeta} f)(\cdot)$ is S-asymptotically (ω, c)-periodic since $|c| < 1$.*

Using the same substitution, we can simply transfer the statements of ([35], Theorem 2, Theorem 5, Theorem 8) and Theorem 7 for the proportional fractional integrals. Details can be left to the interested readers.

In what follows, we continue the general analysis of Section 5. Let (C3) hold. We prove that the solution $u(t)$ of the fractional Cauchy problem (17), which is given by (18), cannot be (ω, c)-periodic provided that $u(t)$ is not a constant multiple of the function $e^{\frac{\zeta-1}{\zeta}t}$, as well as the function $f(t, x)$ is continuous and satisfies

$$f(t+\omega, cx) = cf(t, x), \quad t \geq 0,\ x \in X. \tag{21}$$

Suppose the contrary and consider the function $v(t) := e^{(1-\zeta)t/\zeta}u(t)$, $t \geq 0$. Then, $v(t)$ is a nonconstant ω-periodic function since

$$v(t+\omega) = e^{(1-\zeta)t/\zeta}e^{(1-\zeta)\omega/\zeta}u(t+\omega) = e^{(1-\zeta)t/\zeta}e^{(1-\zeta)\omega/\zeta}cu(t) = v(t), \quad t \geq 0;$$

moreover, we have:

$$v(t) = u_0 + \int_0^t g_\alpha(t-s)\Big[\zeta^{-\alpha}e^{\frac{1-\zeta}{\zeta}s} f(s, u(s))\Big]\, ds, \quad t \geq 0.$$

This implies

$$\mathbf{D}_t^{\alpha} v(t) = \frac{d}{dt}\Big[g_{1-\alpha} * \big(v(\cdot) - u_0\big)\Big] = \zeta^{-\alpha} e^{\frac{1-\zeta}{\zeta}t} f(t, u(t)), \quad t \geq 0.$$

On the other hand, using (21), we have

$$f(t+\omega, u(t+\omega)) = f(t+\omega, cu(t)) = cf(t, u(t)), \quad t \geq 0,$$

so that the mapping $t \mapsto f(t, u(t))$, $t \geq 0$ is (ω, c)-periodic, and consequently, the mapping $t \mapsto \zeta^{-\alpha} e^{\frac{1-\zeta}{\zeta}t} f(t, u(t))$, $t \geq 0$ is ω-periodic. This contradicts ([37], Corollary 2) (see also Remark 1) and yields the required conclusion.

We can similarly prove that the function $e^{(1-\zeta)\cdot/\zeta}u(\cdot)$ cannot be Poisson stable (a restriction of an almost automorphic function to the non-negative real line), which strongly justifies the consideration of our results from Section 4.

6. Conclusions and Final Remarks

In this research article, we reconsidered the notion of the proportional Caputo fractional derivative of order $\alpha \in (0,1)$ and provided a new theoretical concept of the proportional Caputo fractional derivative of order α for the functions that are not continuously differentiable, in general. We investigated the existence and uniqueness of almost periodic-type solutions for various classes of proportional Caputo fractional differential inclusions in Banach spaces and explored the basic properties of the fractional solution operator families connected with the use of this type of fractional derivatives; the considered fractional solution operator families are subgenerated by multivalued linear operators and can be degenerate in the time variable. In addition to the above, several questions, observations and open problems are proposed.

Let us finally note the following:

1. Of concern is the following generalization of (20):

$$\int_0^{\omega} a(t+s) f(\omega - s)\, ds = 0, \quad t \geq 0, \tag{22}$$

with $a \in L^1_{loc}([0,\infty) : \mathbb{C})$ and $f \in L^1_{loc}([0,\omega) : X)$. It could be of importance to find some sufficient conditions on the kernel $a(t)$ ensuring that the assumption (22) implies $f(t) = 0$ for a.e. $t \in [0,\omega]$. For general non-constant kernels $a(t)$, this is not true as the following simple counterexample shows:

Example 3. *Suppose that* $a(t) = f(t) = 1$ *for* $0 \leq t < \omega/2$ *and* $a(t) = f(t) = 0$ *for* $\omega/2 \leq t \leq \omega$. *Then,* (22) *holds but it is not true that* $f(t) = 0$ *for a.e.* $t \in [0,\omega]$; *moreover,* $0 \in supp(a) \cap supp(f)$.

2. In this paper, we did not consider the Caputo fractional proportional derivatives with respect to another functions and the abstract fractional inclusions with this type of fractional derivatives. For more detail on the subject, we refer the reader to the research articles [52–54] and the list of references quoted therein.
3. The Hilfer generalized proportional fractional derivatives have been also introduced and analyzed in the existing literature (see, e.g., the paper [55] by I. Ahmed et al). Concerning the Hadamard proportional fractional integral inequalities, we can recommend for the reader [56–58] and the references cited therein.

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