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# Fejér-Type Inequalities for Harmonically Convex Functions and Related Results

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**Abstract:** In this paper, new Fejér-type inequalities for harmonically convex functions are obtained. Some mappings related to the Fejér-type inequalities for harmonically convex are defined. Properties of these mappings are discussed and, as a consequence, we obtain refinements of some known results.

**Keywords:** Hermite-Hadamard inequality; convex function; harmonically convex function; Fejér inequality

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## 1. Introduction

For convex functions the following double inequality has great significance in the literature and is known as the Hermite-Hadamard's inequality [1,2]:

Let $\chi$: $I \longrightarrow \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$ be a convex function, then

$$\chi\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \chi(\nu) d\nu \leq \frac{\chi(\varkappa_1) + \chi(\varkappa_2)}{2}, \tag{1}$$

for all $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$. The inequality (1) holds in reversed direction if $\chi$ is concave. f Dragomir defined the following mappings $H$, $F$: $[0,1] \to \mathbb{R}$

$$H(\kappa) = \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \chi\left(\kappa\nu + (1-\kappa)\left(\frac{\varkappa_1 + \varkappa_2}{2}\right)\right) d\nu$$

and

$$F(\kappa) = \frac{1}{(\varkappa_2 - \varkappa_1)^2} \int_{\varkappa_1}^{\varkappa_2} \int_{\varkappa_1}^{\varkappa_2} \chi(\kappa\nu + (1-\kappa)\tilde{\omega}) d\nu d\tilde{\omega},$$

where $\chi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a convex function and obtained some refinements between the middle and the left most terms in [3] for (1).

**Theorem 1** ([3]). *Let $\chi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a convex function on $[\varkappa_1, \varkappa_2]$. Then*

*(i)* *$H$ is convex on $[\varkappa_1, \varkappa_2]$.*
*(ii)* *The following hold:*

$$\inf_{\kappa \in [0,1]} H(\kappa) = H(0) = \chi\left(\frac{\varkappa_1 + \varkappa_2}{2}\right)$$

$$\sup_{\kappa \in [0,1]} H(\kappa) = H(1) = \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \chi(\nu) d\nu.$$

*(iii)* *$H$ increases monotonically on $[0,1]$.*

**Theorem 2** ([3]). *Let $\chi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a convex function on $[\varkappa_1, \varkappa_2]$. Then*

*(i)* $F\left(\kappa+\frac{1}{2}\right)=F\left(\frac{1}{2}-\kappa\right)$ *for all* $\kappa\in\left[0,\frac{1}{2}\right]$.
*(ii)* $F$ *is convex on* $[\varkappa_1,\varkappa_2]$.
*(iii)* *The following hold:*

$$\sup_{\kappa\in[0,1]} F(\kappa)=F(1)=\chi(0)=\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\chi(\nu)d\nu$$

*and*

$$\inf_{\kappa\in[0,1]} F(\kappa)=F\left(\frac{1}{2}\right)=\frac{1}{(\varkappa_2-\varkappa_1)^2}\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}F\left(\frac{\nu+\tilde{\omega}}{2}\right)d\nu d\tilde{\omega}.$$

*(iv)* *The inequality*

$$F\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\leq F\left(\frac{1}{2}\right)$$

*is valid.*
*(v)* $\chi$ *increases monotonically on* $\left[\frac{1}{2},1\right]$ *and decreases monotonically on* $\left[0,\frac{1}{2}\right]$.
*(vi)* *We have the inequality* $H(\kappa)\leq F(\kappa)$ *for all* $\kappa\in[0,1]$.

Yang and Hong [4] provided an improvement between the middle and the right-most term by defining the following mapping $P$: $[0,1]\to\mathbb{R}$

$$P(\kappa)=\frac{1}{2(\varkappa_2-\varkappa_1)}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\left(\frac{1+\kappa}{2}\right)\varkappa_2+\left(\frac{1-\kappa}{2}\right)\nu\right)\right.$$
$$\left.+\chi\left(\left(\frac{1+\kappa}{2}\right)\varkappa_1+\left(\frac{1-\kappa}{2}\right)\nu\right)\right]d\nu,$$

where $\chi$: $[\varkappa_1,\varkappa_2]\to\mathbb{R}$ is a convex function.

**Theorem 3** ([4])**.** *Let* $\chi$*:* $[\varkappa_1,\varkappa_2]\to\mathbb{R}$ *be a convex function on* $[\varkappa_1,\varkappa_2]$*. Then*

*(i)* $P$ *is convex on* $[\varkappa_1,\varkappa_2]$.
*(ii)* $P$ *increases monotonically on* $[0,1]$.
*(iii)* *The following hold*

$$\inf_{\kappa\in[0,1]} P(\kappa)=P(0)=\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\chi(\nu)d\nu$$

*and*

$$\sup_{\kappa\in[0,1]} P(\kappa)=P(1)=\frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2}.$$

Fejér [5], established the following double inequality as a weighted generalization of (1):

$$\chi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\int_{\varkappa_1}^{\varkappa_2}\varphi(\nu)d\nu\leq\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\chi(\nu)\varphi(\nu)d\nu\leq\frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\varphi(\nu)d\nu, \quad (2)$$

where $\chi$: $I\longrightarrow\mathbb{R}$, $\emptyset\neq I\subseteq\mathbb{R}$, $\varkappa_1,\varkappa_2\in I$ with $\varkappa_1<\varkappa_2$ is any convex function and $\varphi$: $[\varkappa_1,\varkappa_2]\to\mathbb{R}$ is a non-negative integrable with $\varphi(\varkappa_1+\varkappa_2-\nu)=\varphi(\nu)$.

These inequalities have many extensions and generalizations, see [6–34].

Teseng et al. [35] refined inequalities (2) by defining the following mappings on $[0,1]$:

$$I(\kappa)=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\kappa\frac{\nu+\varkappa_1}{2}+(1-\kappa)\frac{\varkappa_1+\varkappa_2}{2}\right)+\chi\left(\kappa\frac{\nu+\varkappa_2}{2}+(1-\kappa)\frac{\varkappa_1+\varkappa_2}{2}\right)\right]\varphi(\nu)d\nu,$$

$$J(\kappa)=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\kappa\frac{\nu+\varkappa_1}{2}+(1-\kappa)\frac{3\varkappa_1+\varkappa_2}{4}\right)+\chi\left(\kappa\frac{\nu+\varkappa_2}{2}+(1-\kappa)\frac{\varkappa_1+3\varkappa_2}{4}\right)\right]\varphi(\nu)d\nu,$$

$$\begin{aligned}M(\kappa)=&\frac{1}{2}\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}\left[\chi\left(\kappa\varkappa_1+(1-\kappa)\frac{\nu+\varkappa_1}{2}\right)+\chi\left(\kappa\frac{\varkappa_1+\varkappa_2}{2}+(1-\kappa)\frac{\nu+\varkappa_2}{2}\right)\right]\varphi(\nu)d\nu\\&+\frac{1}{2}\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}\left[\chi\left(\kappa\frac{\varkappa_1+\varkappa_2}{2}+(1-\kappa)\frac{\nu+\varkappa_1}{2}\right)+\chi\left(\kappa\varkappa_2+(1-\kappa)\frac{\nu+\varkappa_2}{2}\right)\right]\varphi(\nu)d\nu\end{aligned}$$

and

$$N(\kappa)=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\kappa\varkappa_1+(1-\kappa)\frac{\nu+\varkappa_1}{2}\right)+\chi\left(\kappa\varkappa_2+(1-\kappa)\frac{\nu+\varkappa_2}{2}\right)\right]\varphi(\nu)d\nu,$$

where $\chi$: $[\varkappa_1,\varkappa_2]\to\mathbb{R}$ is a convex function and $\varphi$: $[\varkappa_1,\varkappa_2]\to\mathbb{R}$ is a non-negative integrable with $\varphi(\varkappa_1+\varkappa_2-\nu)=\varphi(\nu)$.

By applying the result given below,

**Lemma 1** ([35])**.** *Let $\chi:[\varkappa_1,\varkappa_2]\to\mathbb{R}$ be a convex function and let $\varkappa_1\le\tilde{\omega}_1\le\nu_1\le\nu_2\le\tilde{\omega}_2\le\varkappa_2$ with $\nu_1+\nu_2=\tilde{\omega}_1+\tilde{\omega}_2$. Then*

$$\chi(\nu_1)+\chi(\nu_2)\le\chi(\tilde{\omega}_1)+\chi(\tilde{\omega}_2).$$

Teseng et al. obtained the following important refinement inequalities.

**Theorem 4** ([35])**.** *Let $\chi$: $[\varkappa_1,\varkappa_2]\to\mathbb{R}$ be a convex function and $\varphi$: $[\varkappa_1,\varkappa_2]\to\mathbb{R}$ is a non-negative integrable with $\varphi(\varkappa_1+\varkappa_2-\nu)=\varphi(\nu)$. Then I is convex, increasing on $[0,1]$, and for all $\kappa\in[0,1]$, the Fejér-type inequalities*

$$\begin{aligned}\chi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\int_{\varkappa_1}^{\varkappa_2}\varphi(\nu)d\nu=I(0)\le I(\kappa)\le I(1)\\=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{\varkappa_1+\nu}{2}\right)+\chi\left(\frac{\nu+\varkappa_2}{2}\right)\right]\varphi(\nu)d\nu\end{aligned}$$

*hold.*

**Theorem 5** ([35])**.** *Let $\chi$: $[\varkappa_1,\varkappa_2]\to\mathbb{R}$ be a convex function and $\varphi$: $[\varkappa_1,\varkappa_2]\to\mathbb{R}$ is a non-negative integrable with $\varphi(\varkappa_1+\varkappa_2-\nu)=\varphi(\nu)$. Then J is convex, increasing on $[0,1]$, and for all $\kappa\in[0,1]$, the Fejér-type inequalities*

$$\begin{aligned}\frac{\chi\left(\frac{3\varkappa_1+\varkappa_2}{4}\right)+\chi\left(\frac{\varkappa_1+3\varkappa_2}{4}\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\varphi(\nu)d\nu=J(0)\le J(\kappa)\le J(1)\\=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{\varkappa_1+\nu}{2}\right)+\chi\left(\frac{\nu+\varkappa_2}{2}\right)\right]\varphi(\nu)d\nu\end{aligned}$$

*hold.*

**Theorem 6** ([35])**.** *Let $\chi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a convex function and $\varphi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a non-negative integrable with $\varphi(\varkappa_1 + \varkappa_2 - \nu) = \varphi(\nu)$. Then $I(\kappa) \leq J(\kappa)$ on $[0,1]$.*

**Theorem 7** ([35])**.** *Let $\chi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a convex function and $\varphi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a non-negative integrable with $\varphi(\varkappa_1 + \varkappa_2 - \nu) = \varphi(\nu)$. Then $M$ is convex, increasing on $[0,1]$, and for all $\kappa \in [0,1]$, the Fejér-type inequalities*

$$\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{\varkappa_1+\nu}{2}\right)+\chi\left(\frac{\nu+\varkappa_2}{2}\right)\right]\varphi(\nu)d\nu = M(0) \leq M(\kappa)$$

$$\leq M(1) = \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2}\right]\varphi(\nu)d\nu$$

*hold valid.*

**Theorem 8** ([35])**.** *Let $\chi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a convex function and $\varphi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a non-negative integrable with $\varphi(\varkappa_1 + \varkappa_2 - \nu) = \varphi(\nu)$. Then $M$ is convex, increasing on $[0,1]$, and for all $\kappa \in [0,1]$, the Fejér-type inequalities*

$$\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{\varkappa_1+\nu}{2}\right)+\chi\left(\frac{\nu+\varkappa_2}{2}\right)\right]\varphi(\nu)d\nu = N(0)$$

$$\leq N(\kappa) \leq N(1) \leq \frac{\chi(\varkappa_1)+(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\varphi(\nu)d\nu$$

*hold true.*

**Theorem 9** ([35])**.** *Let $\chi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a convex function and $\varphi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a non-negative integrable with $\varphi(\varkappa_1 + \varkappa_2 - \nu) = \varphi(\nu)$. Then $M(\kappa) \leq N(\kappa)$ on $[0,1]$.*

One of the generalizations of the convex functions is harmonic functions:

**Definition 1** ([36])**.** *Define $I \subseteq \mathbb{R}\backslash\{0\}$ as an interval of real numbers. We say that a function $\chi$ from $I$ to $\mathbb{R}$ is considered to be harmonically convex, if*

$$\chi\left(\frac{\nu\tilde{\omega}}{\kappa\nu+(1-\kappa)\tilde{\omega}}\right) \leq \kappa\chi(\tilde{\omega}) + (1-\kappa)\chi(\nu) \tag{3}$$

*for all $\nu, \tilde{\omega} \in I$ and $\kappa \in [0,1]$. Harmonically concave $\chi$ is defined as the inequality in (3) reversed.*

Using harmonic-convexity, the Hermite-Hadamard type yields the following result.

**Theorem 10** ([36])**.** *Let $\chi$: $I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$. If $\chi \in L([\varkappa_1, \varkappa_2])$, then the inequalities*

$$\chi\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right) \leq \frac{\varkappa_1\varkappa_2}{\varkappa_2-\varkappa_1}\int_{\varkappa_2}^{\varkappa_1}\frac{\chi(\nu)}{\nu^2}d\nu \leq \frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2} \tag{4}$$

*hold.*

Let $\chi\colon [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a harmonically convex mapping and let $S, U, V\colon [0,1] \to \mathbb{R}$ be defined by

$$S(\kappa) = \frac{\varkappa_1 \varkappa_2}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu^2} \chi\left( \frac{2\varkappa_1 \varkappa_2 \nu}{2\varkappa_1 \varkappa_2 \kappa + (1-\kappa)\nu(\varkappa_1 + \varkappa_2)} \right) d\nu, \tag{5}$$

$$U(\kappa) = \left( \frac{\varkappa_1 \varkappa_2}{\varkappa_2 - \varkappa_1} \right)^2 \int_{\varkappa_1}^{\varkappa_2} \int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu^2 \tilde{\omega}^2} \chi\left( \frac{\nu \tilde{\omega}}{\kappa \tilde{\omega} + (1-\kappa)\nu} \right) d\nu d\tilde{\omega} \tag{6}$$

and

$$\begin{aligned} V(\kappa) = \frac{\varkappa_1 \varkappa_2}{2(\varkappa_2 - \varkappa_1)} \int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu^2} \bigg[ &\chi\left( \frac{2\varkappa_2 \nu}{(1+\kappa)\nu + (1-\kappa)\varkappa_2} \right) \\ &+ \chi\left( \frac{2\varkappa_1 \nu}{(1+\kappa)\nu + (1-\kappa)\varkappa_1} \right) \bigg] d\nu. \end{aligned} \tag{7}$$

The author obtained the refinement inequalities for (4) corresponding to the above mappings:

**Theorem 11** ([23])**.** *Let $\chi\colon [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a harmonically convex function on $[\varkappa_1, \varkappa_2]$. Then*

*(i)* *$S$ is harmonically convex on $[\varkappa_1, \varkappa_2]$.*

*(ii)* *The following hold:*

$$\inf_{\kappa \in [0,1]} S(\kappa) = S(0) = \chi\left( \frac{2\varkappa_1 \varkappa_2}{\varkappa_1 + \varkappa_2} \right)$$

*and*

$$\sup_{\kappa \in [0,1]} S(\kappa) = S(1) = \frac{\varkappa_1 \varkappa_2}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \frac{\chi(\nu)}{\nu^2} d\nu.$$

*(iii)* *$S$ increases monotonically on $[0,1]$.*

**Theorem 12** ([23])**.** *Let $\chi\colon [\varkappa_1, \varkappa_2] \subset (0, \infty) \to \mathbb{R}$ be a harmonically convex function on $[\varkappa_1, \varkappa_2]$. Then*

*(i)* *The identity*

$$U\left( \kappa + \frac{1}{2} \right) = U\left( \frac{1}{2} - \hat{\kappa} \right)$$

*holds for all $\kappa \in \left[0, \frac{1}{2}\right]$.*

*(ii)* *$U$ is harmonically convex on $[\varkappa_1, \varkappa_2]$.*

*(iii)* *The identities*

$$\inf_{\kappa \in [0,1]} U(\kappa) = U\left( \frac{1}{2} \right) = \left( \frac{\varkappa_1 \varkappa_2}{\varkappa_2 - \varkappa_1} \right)^2 \int_{\varkappa_1}^{\varkappa_2} \int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu^2 \tilde{\omega}^2} \chi\left( \frac{2\nu \tilde{\omega}}{\nu + \tilde{\omega}} \right) d\nu d\tilde{\omega}$$

*and*

$$\sup_{\kappa \in [0,1]} U(\kappa) = U(0) = U(1) = \frac{\varkappa_1 \varkappa_2}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \frac{\chi(\nu)}{\nu^2} d\nu$$

*hold.*

*(iv)* *The inequality*

$$\chi\left( \frac{2\varkappa_1 \varkappa_2}{\varkappa_1 + \varkappa_2} \right) \leq U\left( \frac{1}{2} \right)$$

*holds true.*

*(v)* *$U$ increases monotonically on $\left[\frac{1}{2}, 1\right]$ and decreases monotonically on $\left[0, \frac{1}{2}\right]$.*

*(vi)* *$S(\kappa) \leq U(\kappa)$ for all $\kappa \in [0,1]$.*

**Theorem 13** ([23]). *Let $\chi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a harmonically convex function on $[\varkappa_1, \varkappa_2]$. Then*

*(i)* *$V$ is harmonically convex on $[\varkappa_1, \varkappa_2]$.*

*(ii)* *The following hold:*

$$\inf_{\kappa\in[0,1]} V(\kappa) = V(0) = \frac{\varkappa_1\varkappa_2}{\varkappa_2 - \varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \frac{\chi(\nu)}{\nu^2}d\nu$$

*and*

$$\sup_{\kappa\in[0,1]} V(\kappa) = V(1) = \frac{\chi(\varkappa_1) + \chi(\varkappa_2)}{2}.$$

*(iii)* *$V$ increases monotonically on $[0, 1]$.*

Harmonic symmetricity of a function is given in the definition below.

**Definition 2** ([24]). *A function $\varphi$: $[\varkappa_1, \varkappa_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is harmonically symmetric with respect to $\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}$ if*

$$\varphi(\nu) = \varphi\left(\frac{1}{\frac{1}{\varkappa_1} + \frac{1}{\varkappa_2} - \frac{1}{\nu}}\right)$$

*holds for all $\nu \in [\varkappa_1, \varkappa_2]$.*

Fejér type inequalities using harmonic convexity and the notion of harmonic symmetricity were presented in Chan and Wu [25].

**Theorem 14** ([25]). *Let $\chi$: $I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$. If $\chi \in L([\varkappa_1, \varkappa_2])$ and $\varphi$: $[\varkappa_1, \varkappa_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is non-negative, integrable and harmonically symmetric with respect to $\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}$, then*

$$\chi\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right)\int_{\varkappa_2}^{\varkappa_1}\frac{\varphi(\nu)}{\nu^2}d\nu \;\leq\; \int_{\varkappa_2}^{\varkappa_1}\frac{\chi(\nu)\varphi(\nu)}{\nu^2}d\nu \;\leq\; \frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2}\int_{\varkappa_2}^{\varkappa_1}\frac{\varphi(\nu)}{\nu^2}d\nu. \tag{8}$$

Chan and Wu [25] also defined some mappings related to (8) and discussed important properties of these mappings.

Motivated by the studies conducted in [3,4,23,35], we define some new mappings in connection to (8) and prove new Féjer type inequalities, which indeed provide refinement inequalities as well.

## 2. Main Results

We state some important facts which relate harmonically convex and convex functions and use them to prove the main results of this paper.

**Theorem 15** ([26,27]). *If $[\varkappa_1, \varkappa_2] \subset I \subset (0, \infty)$ and if we consider the function $g$: $\left[\frac{1}{\varkappa_2}, \frac{1}{\varkappa_1}\right] \to \mathbb{R}$ defined by $g(\kappa) = \chi\left(\frac{1}{\kappa}\right)$, then $\chi$ is harmonically convex on $[\varkappa_1, \varkappa_2]$, if and only if $g$ is convex in the usual sense on $\left[\frac{1}{\varkappa_2}, \frac{1}{\varkappa_1}\right]$.*

**Theorem 16** ([26,27]). *If $I \subset (0, \infty)$ and $\chi$ is a convex and non-decreasing function, then $\chi$ is HA-convex and if $\chi$ is a HA-convex and non-increasing function, then $\chi$ is convex.*

**Theorem 17** ([26,27]). *Let $\chi$: $I \subset (0, \infty) \to \mathbb{R}$ be a HA-convex function and $[k, K] \subset I^\circ$. Let $\nu : \Omega \to \mathbb{R}$ be satisfying the bounds*

$$0 < k \leq \nu(\kappa) \leq K < \infty \text{ for } \mu\text{-a.e. } \kappa \in \Omega$$

*and $w \geq 0$ $\mu$-a.e. on $\Omega$ with $\int_\Omega w d\mu = 1$. If $\chi \circ \nu$, $\frac{1}{\nu} \in L_w(\Omega, \mu)$, then*

$$\chi\left(\frac{1}{\int_\Omega \frac{w}{\nu} d\mu}\right) \leq \int_\Omega (\chi \circ \nu) w d\mu.$$

Let us now define some mappings on $[0,1]$ related to (8) and prove some refinement inequalities.

$$I_1(\kappa) = \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{\kappa\varkappa_2(\varkappa_1+\nu)+(1-\kappa)\nu(\varkappa_1+\varkappa_2)}\right)\right.$$
$$\left.+\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{\kappa\varkappa_1(\nu+\varkappa_2)+(1-\kappa)\nu(\varkappa_1+\varkappa_2)}\right)\right]\frac{\varphi(\nu)}{\nu^2}d\nu,$$

$$J_1(\kappa) = \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{2\kappa\varkappa_1(\varkappa_2+\nu)+(1-\kappa)\nu(3\varkappa_1+\varkappa_2)}\right)\right.$$
$$\left.+\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{2\kappa\varkappa_2(\varkappa_1+\nu)+(1-\kappa)\nu(3\varkappa_2+\varkappa_1)}\right)\right]\frac{\varphi(\nu)}{\nu^2}d\nu,$$

$$M_1(\kappa) = \frac{1}{2}\int_{\varkappa_1}^{\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}}\left[\chi\left(\frac{2\varkappa_2\nu}{2\nu\kappa+(1-\kappa)(\nu+\varkappa_2)}\right)\right.$$
$$\left.+\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{\kappa\nu(\varkappa_1+\varkappa_2)+(1-\kappa)\varkappa_2(\varkappa_1+\nu)}\right)\right]\frac{\varphi(\nu)}{\nu^2}d\nu$$
$$+\frac{1}{2}\int_{\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}}^{\varkappa_2}\left[\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{\kappa\nu(\varkappa_1+\varkappa_2)+(1-\kappa)\varkappa_1(\varkappa_2+\nu)}\right)\right.$$
$$\left.+\chi\left(\frac{2\varkappa_1\nu}{2\nu\kappa+(1-\kappa)(\nu+\varkappa_1)}\right)\right]\frac{\varphi(\nu)}{\nu^2}d\nu$$

and

$$N_1(\kappa) = \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{2\varkappa_2\nu}{2\nu\kappa+(1-\kappa)(\nu+\varkappa_2)}\right)+\chi\left(\frac{2\varkappa_1\nu}{2\nu\kappa+(1-\kappa)(\varkappa_1+\nu)}\right)\right]\frac{\varphi(\nu)}{\nu^2}d\nu,$$

where $\chi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a harmonically convex function and $\varphi$: $[\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a non-negative integrable and symmetric about $\nu = \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}$.

**Lemma 2** ([28])**.** *Let $\chi : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a harmonically convex function and let $\varkappa_1 \leq \tilde{\omega}_1 \leq \nu_1 \leq \nu_2 \leq \tilde{\omega}_2 \leq \varkappa_2$ with $\frac{\nu_1\nu_2}{\nu_1+\nu_2} = \frac{\tilde{\omega}_1\tilde{\omega}_2}{\tilde{\omega}_1+\tilde{\omega}_2}$. Then*

$$\chi(\nu_1) + \chi(\nu_2) \leq \chi(\tilde{\omega}_1) + \chi(\tilde{\omega}_2).$$

**Theorem 18.** *Let $\chi, \varphi, I_1$ be defined as above. Then $I_1$ is harmonically convex, increasing on $[0,1]$ and the Fejér-type inequalities*

$$\chi\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\varphi(\nu)}{\nu^2}d\nu = I_1(0) \leq I_1(\kappa)$$
$$\leq I_1(1) = \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{2\varkappa_1\nu}{\varkappa_1+\nu}\right)+\chi\left(\frac{2\varkappa_2\nu}{\nu+\varkappa_2}\right)\right]\frac{\varphi(\nu)}{\nu^2}d\nu \quad (9)$$

*hold for all* $\kappa \in [0,1]$.

**Proof.** The mapping $I_1 : [0,1] \to \mathbb{R}$ is harmonically convex if and only if the mapping $\bar{I}_1 : [0,1] \to \mathbb{R}$ defined by

$$\bar{I}_1(\kappa) = \frac{1}{2}\int_{\frac{1}{\varkappa_2}}^{\frac{1}{\varkappa_1}} \left[ g\left(\kappa\left(\frac{\varkappa_1+\nu}{2\varkappa_1\nu}\right) + (1-\kappa)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right) + g\left(\kappa\left(\frac{\nu+\varkappa_2}{2\varkappa_2\nu}\right) + (1-\kappa)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right]\varphi\left(\frac{1}{\nu}\right)d\nu$$

is convex for a convex mapping $g$: $\left[\frac{1}{\varkappa_2}, \frac{1}{\varkappa_1}\right] \to \mathbb{R}$. Let $\kappa_1, \kappa_2 \in [0,1]$, $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, then

$$\begin{aligned}
\bar{I}_1(\kappa_1\alpha + \kappa_2\beta) &= \frac{1}{2}\int_{\frac{1}{\varkappa_2}}^{\frac{1}{\varkappa_1}} \left[ g\left((\kappa_1\alpha+\kappa_2\beta)\left(\frac{\varkappa_1+\nu}{2\varkappa_1\nu}\right) + (1-(\kappa_1\alpha+\kappa_2\beta))\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right. \\
&\quad \left. + g\left((\kappa_1\alpha+\kappa_2\beta)\left(\frac{\nu+\varkappa_2}{2\varkappa_2\nu}\right) + (1-(\kappa_1\alpha+\kappa_2\beta))\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right]\varphi\left(\frac{1}{\nu}\right)d\nu \\
&= \frac{1}{2}\int_{\frac{1}{\varkappa_2}}^{\frac{1}{\varkappa_1}} \left[ g\left(\alpha\left(\kappa_1\left(\frac{\varkappa_1+\nu}{2\varkappa_1\nu}\right) + (1-\kappa_1)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right.\right. \\
&\quad \left. + \beta\left(\kappa_2\left(\frac{\varkappa_1+\nu}{2\varkappa_1\nu}\right) + (1-\kappa_2)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right) + g\left(\alpha\left(\kappa_1\left(\frac{\varkappa_1+\nu}{2\varkappa_2\nu}\right) + (1-\hat{\kappa}_1)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right. \\
&\quad \left.\left. + \beta\left(\kappa_2\left(\frac{\varkappa_1+\nu}{2\varkappa_1\nu}\right) + (1-\kappa_2)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right)\right]\varphi\left(\frac{1}{\nu}\right)d\nu \\
&\leq \alpha\frac{1}{2}\int_{\frac{1}{\varkappa_2}}^{\frac{1}{\varkappa_1}} \left[ g\left(\kappa_1\left(\frac{\varkappa_1+\nu}{2\varkappa_1\nu}\right) + (1-\kappa_1)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right. \\
&\quad \left. + g\left(\kappa_1\left(\frac{\varkappa_1+\nu}{2\varkappa_2\nu}\right) + (1-\kappa_1)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right]\varphi\left(\frac{1}{\nu}\right)d\nu \\
&\quad + \beta\frac{1}{2}\int_{\frac{1}{\varkappa_2}}^{\frac{1}{\varkappa_1}} \left[ g\left(\kappa_2\left(\frac{\varkappa_1+\nu}{2\varkappa_2\nu}\right) + (1-\kappa_2)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right. \\
&\quad \left. + g\left(\kappa_2\left(\frac{\varkappa_1+\nu}{2\varkappa_2\nu}\right) + (1-\kappa_2)\left(\frac{\varkappa_1+\varkappa_2}{2\varkappa_1\varkappa_2}\right)\right)\right]\varphi\left(\frac{1}{\nu}\right)d\nu = \alpha\bar{I}_1(\kappa_1) + \beta\bar{I}_1(\kappa_2).
\end{aligned}$$

This proves the harmonic convexity of $I_1$: $[0,1] \to \mathbb{R}$.
By integrating and making the following assumptions on $\varphi$, the following identity is true for [0,1]:

$$I_1(\kappa) = \int_{\varkappa_1}^{\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}} \left[ \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa\varkappa_1\varkappa_2 + (1-\kappa)\nu(\varkappa_1+\varkappa_2)}\right) + \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa((\varkappa_1+\varkappa_2)\nu - \varkappa_1\varkappa_2) + (1-\kappa)(\varkappa_1+\varkappa_2)\nu}\right)\right]\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2}d\nu. \quad (10)$$

Let $\kappa_1, \kappa_2 \in [0,1]$ with $\kappa_1 < \kappa_2$. Choosing

$$\begin{aligned}\nu_1 &= \frac{2\varkappa_1\varkappa_2\nu}{2\kappa_2\varkappa_1\varkappa_2 + (1-\kappa_2)\nu(\varkappa_1+\varkappa_2)},\\ \nu_2 &= \frac{2\varkappa_1\varkappa_2\nu}{2\kappa_2((\varkappa_1+\varkappa_2)\nu - \varkappa_1\varkappa_2) + (1-\kappa_2)(\varkappa_1+\varkappa_2)\nu},\\ \tilde{\omega}_1 &= \frac{2\varkappa_1\varkappa_2\nu}{2\hat{\kappa}_1\varkappa_1\varkappa_2 + (1-\kappa_1)\nu(\varkappa_1+\varkappa_2)}\end{aligned}$$

and

$$\tilde{\omega}_2 = \frac{2\varkappa_1\varkappa_2\nu}{2\kappa_1((\varkappa_1+\varkappa_2)\nu - \varkappa_1\varkappa_2) + (1-\kappa_1)(\varkappa_1+\varkappa_2)\nu}.$$

Hence, according to Lemma 2, the inequality

$$\begin{aligned}&\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa_1\varkappa_1\varkappa_2 + (1-\kappa_1)\nu(\varkappa_1+\varkappa_2)}\right)\\ &\qquad+\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa_1((\varkappa_1+\varkappa_2)\nu - \varkappa_1\varkappa_2) + (1-\kappa_1)(\varkappa_1+\varkappa_2)\nu}\right)\\ &\qquad\quad\le \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa_2\varkappa_1\varkappa_2 + (1-\kappa_2)\nu(\varkappa_1+\varkappa_2)}\right)\\ &\qquad\qquad+\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa_2((\varkappa_1+\varkappa_2)\nu - \varkappa_1\varkappa_2) + (1-\kappa_2)(\varkappa_1+\varkappa_2)\nu}\right)\end{aligned} \tag{11}$$

holds for all $\nu \in \left[\varkappa_1, \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right]$.
Multiplying (11) by $\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)$, integrating both sides over $\nu$ on $\left[\varkappa_1, \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right]$ and using (10), we derive $I_1(\kappa_1) \le I_1(\kappa_2)$. Thus, $I_1$ is increasing on $[0,1]$ and then the inequality (9) holds. □

**Example 1.** *Let $\chi(\nu) = \nu^{-2}$, $\nu \in [1,2]$, then $\chi$ is harmonically convex. Let $\varphi(\nu) = \left(\frac{1}{\nu} - \frac{3}{4}\right)^2$, $\nu \in [1,2]$. It is clear that $\varphi$ is harmonically symmetric with respect to $\frac{4}{3}$. By using the techniques of integration, we observed the following calculations for $0 \le \kappa \le 1$:*

$$I_1(0) = \left(\frac{3}{4}\right)^2 \int_1^2 \frac{1}{\nu^2}\left(\frac{1}{\nu} - \frac{3}{4}\right)^2 d\nu = \frac{3}{512},$$

$$\begin{aligned}I_1(\kappa) = \frac{1}{2}\int_1^2 \frac{1}{\nu^2}\left(\frac{1}{\nu}-\frac{3}{4}\right)^2 &\left[\left(\frac{4\nu}{2\kappa(\nu+1)+3(1-\kappa)\nu}\right)^{-2}\right.\\ &\left.+\left(\frac{4\nu}{\kappa(\nu+2)+3(1-\kappa)\nu}\right)^{-2}\right] d\nu = \frac{2\kappa^2+45}{7680},\end{aligned}$$

*and*

$$I_1(1) = \frac{1}{2}\int_1^2 \frac{1}{\nu^2}\left(\left(\frac{4\nu}{\nu+2}\right)^{-2} + \left(\frac{2\nu}{\nu+1}\right)^{-2}\right)\left(\frac{1}{\nu}-\frac{3}{4}\right)^2 d\nu = \frac{47}{7680}.$$

*The Figure 1 below validates the inequality (9) in Theorem 18.*

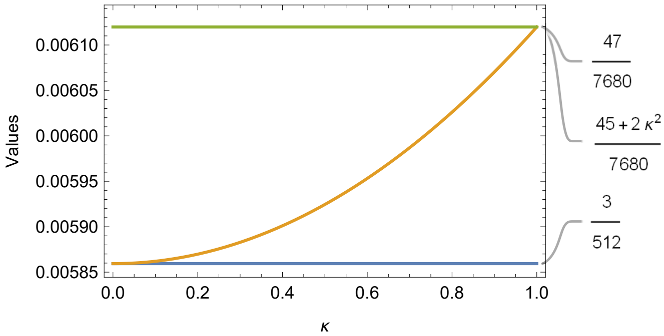


**Figure 1.** The graph of inequality (9) for $0 \leq \kappa \leq 1$.

**Remark 1.** *Let* $\varphi(\nu) = \frac{\varkappa_1\varkappa_2}{\varkappa_2-\varkappa_1}$, $\nu \in [\varkappa_1, \varkappa_2]$ *in Theorem 18. Then* $I_1(\kappa) = S(\kappa)$, $\kappa \in [0,1]$ *and the inequalities (9) take the form*

$$\chi\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right) = S(0) \leq S(\kappa) \leq S(1) = \frac{\varkappa_1\varkappa_2}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\chi(\nu)}{\nu^2}d\nu,$$

*where S is defined by (5).*

**Theorem 19.** *Let* $\chi, \varphi, J_1$ *be defined as above. Then* $J_1$ *is harmonically convex, increasing on* $[0,1]$ *and the Fejér-type inequalities*

$$\frac{\chi\left(\frac{4\varkappa_1\varkappa_2}{\varkappa_1+3\varkappa_2}\right)+\chi\left(\frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{\varphi(\nu)}{\nu^2}d\nu \leq J_1(0) \leq J_1(\kappa) \leq J_1(1)$$
$$= \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{2\varkappa_1\nu}{\varkappa_1+\nu}\right)+\chi\left(\frac{2\varkappa_2\nu}{\nu+\varkappa_2}\right)\right]\frac{\varphi(\nu)}{\nu^2}d\nu \quad (12)$$

*hold for all* $\kappa \in [0,1]$.

**Proof.** The harmonic convexity of $J_1$ on $[0,1]$ can be proved similarly as in proving the harmonic convexity of $I_1$ on $[0,1]$.
The following identity holds on $[0,1]$:

$$\begin{aligned} J_1(\kappa) = \int_{\varkappa_1}^{\frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}} \Bigg[ & \chi\left(\frac{4\varkappa_1\varkappa_2\nu}{4\kappa\varkappa_1\varkappa_2+(1-\kappa)(3\varkappa_1+\varkappa_2)\nu}\right) \\ & +\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa)(3\varkappa_1+\varkappa_2)\nu+\kappa(2(3\varkappa_1+\varkappa_2)\nu-4\varkappa_1\varkappa_2)}\right) \\ & +\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa)(3\varkappa_1+\varkappa_2)\nu+\kappa(4\varkappa_1\varkappa_2+2(\varkappa_2-\varkappa_1)\nu)}\right) \\ & +\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa)(\varkappa_1+3\varkappa_2)\nu+4\kappa((\varkappa_1+\varkappa_2)\nu-\varkappa_1\varkappa_2)}\right)\Bigg]\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2}d\nu. \end{aligned} \quad (13)$$

Let $\kappa_1, \kappa_2 \in [0,1]$ with $\kappa_1 < \kappa_2$. Choosing

$$\begin{aligned} \nu_1 &= \frac{4\varkappa_1\varkappa_2\nu}{4\kappa_2\varkappa_1\varkappa_2+(1-\kappa_2)(3\varkappa_1+\varkappa_2)\nu}, \\ \nu_2 &= \frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_2)(3\varkappa_1+\varkappa_2)\nu+\kappa_2(2(3\varkappa_1+\varkappa_2)\nu-4\varkappa_1\varkappa_2)}, \\ \tilde{\omega}_1 &= \frac{4\varkappa_1\varkappa_2\nu}{4\hat{\kappa}_1\varkappa_1\varkappa_2+(1-\kappa_1)(3\varkappa_1+\varkappa_2)\nu} \end{aligned}$$

and

$$\tilde{\omega}_2 = \frac{4\varkappa_1\varkappa_2\nu}{(1-\hat{\kappa}_1)(3\varkappa_1+\varkappa_2)\nu+\kappa_{12}(2(3\varkappa_1+\varkappa_2)\nu-4\varkappa_1\varkappa_2)}.$$

By using Lemma 2, we obtain the following inequality:

$$\begin{aligned}\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{4\kappa_2\varkappa_1\varkappa_2+(1-\kappa_2)(3\varkappa_1+\varkappa_2)\nu}\right)\\ +\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_2)(3\varkappa_1+\varkappa_2)\nu+\kappa_2(2(3\varkappa_1+\varkappa_2)\nu-4\varkappa_1\varkappa_2)}\right)\\ \leq \chi\left(\frac{4\varkappa_1\varkappa_2\nu}{4\kappa_1\varkappa_1\varkappa_2+(1-\kappa_1)(3\varkappa_1+\varkappa_2)\nu}\right)\\ +\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_1)(3\varkappa_1+\varkappa_2)\nu+\kappa_1(2(3\varkappa_1+\varkappa_2)\nu-4\varkappa_1\varkappa_2)}\right)\end{aligned} \tag{14}$$

for all $\nu \in \left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right]$.
In a similar way, with the choices

$$\begin{aligned}\nu_1 &= \frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_2)(3\varkappa_1+\varkappa_2)\nu+\kappa_2(4\varkappa_1\varkappa_2+2(\varkappa_2-\varkappa_1)\nu)},\\ \nu_2 &= \frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_2)(\varkappa_1+3\varkappa_2)\nu+4\hat{\kappa}_2(\varkappa_1\nu+\varkappa_2\nu-\varkappa_1\varkappa_2)},\\ \tilde{\omega}_1 &= \frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_1)(3\varkappa_1+\varkappa_2)\nu+\kappa_1(4\varkappa_1\varkappa_2+2(\varkappa_2-\varkappa_1)\nu)}\end{aligned}$$

and

$$\tilde{\omega}_2 = \frac{4\varkappa_1\varkappa_2\nu}{(1-\hat{\kappa}_1)(\varkappa_1+3\varkappa_2)\nu+4\kappa_1(\varkappa_1\nu+\varkappa_2\nu-\varkappa_1\varkappa_2)}$$

for $\kappa_1, \kappa_2 \in [0,1]$, where $\kappa_1 < \kappa_2$ and using Lemma 2, we obtain

$$\begin{aligned}\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_2)(3\varkappa_1+\varkappa_2)\nu+\kappa_2(4\varkappa_1\varkappa_2+2(\varkappa_2-\varkappa_1)\nu)}\right)\\ +\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_2)(\varkappa_1+3\varkappa_2)\nu+4\kappa_2(\varkappa_1\nu+\varkappa_2\nu-\varkappa_1\varkappa_2)}\right)\\ \leq \chi\left(\frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_1)(3\varkappa_1+\varkappa_2)\nu+\kappa_1(4\varkappa_1\varkappa_2+2(\varkappa_2-\varkappa_1)\nu)}\right)\\ +\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{(1-\kappa_1)(\varkappa_1+3\varkappa_2)\nu+4\kappa_1(\varkappa_1\nu+\varkappa_2\nu-\varkappa_1\varkappa_2)}\right),\end{aligned} \tag{15}$$

where for all $\nu \in \left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right]$.

Adding (14) and (15), multiplying both sides by $\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2}$ and then integrating over $\left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right]$, we get that $J_1(\kappa_1) \leq J_1(\kappa_2)$ for $\kappa_1, \kappa_2 \in [0,1]$, where $\kappa_1 < \kappa_2$. It is proved that $J_1$ is increasing on $[0,1]$ and hence the inequality (12) is proved because of the fact that $J_1(0) \leq J_1(\kappa) \leq J_1(1)$. $\square$

**Example 2.** *Let $\chi(\nu) = \nu^{-2}$, $\nu \in [1,2]$, then $\chi$ is harmonically convex. Let $\varphi(\nu) = \left(\frac{1}{\nu} - \frac{3}{4}\right)^2$, $\nu \in [1,2]$. It is clear that $\varphi$ is harmonically symmetric with respect to $\frac{4}{3}$. By using the techniques of integration, we observed the following calculations for $0 \leq \kappa \leq 1$:*

$$J_1(0) = \frac{1}{2}\left[\left(\frac{8}{7}\right)^{-2} + \left(\frac{8}{5}\right)^{-2}\right]\int_1^2 \frac{1}{\nu^2}\left(\frac{1}{\nu} - \frac{3}{4}\right)^2 d\nu = \frac{37}{6144},$$

$$J_1(\kappa) = \frac{1}{2}\int_1^2 \frac{1}{\nu^2}\left(\frac{1}{\nu} - \frac{3}{4}\right)^2 \left[\left(\frac{8\nu}{4\kappa(\nu+1) + 7(1-\kappa)\nu}\right)^{-2} + \left(\frac{8\nu}{2\kappa(\nu+2) + 5(1-\kappa)\nu}\right)^{-2}\right] d\nu = \frac{3\kappa^2 + 185}{30720},$$

*and*

$$J_1(1) = \frac{1}{2}\int_1^2 \frac{1}{\nu^2}\left[\left(\frac{4\nu}{\nu+2}\right)^{-2} + \left(\frac{2\nu}{\nu+1}\right)^{-2}\right]\left(\frac{1}{\nu} - \frac{3}{4}\right)^2 d\nu = \frac{47}{7680}.$$

*The Figure 2 below validates the inequality (9) in Theorem 19.*

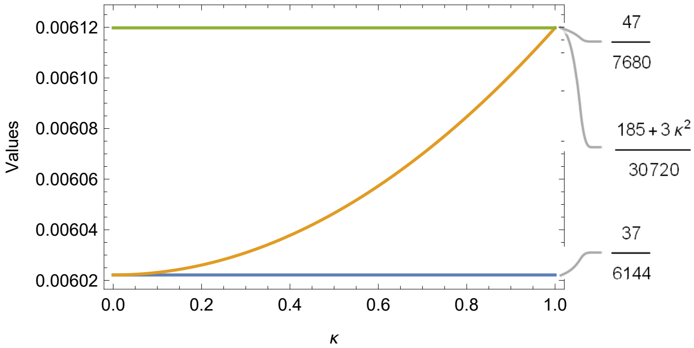


**Figure 2.** The graph of inequality (12) for $0 \leq \kappa \leq 1$.

A comparison between $I_1$ and $J_1$ is given in the theorem below:

**Theorem 20.** *Let $\chi$, $\varphi$, $I_1$, $J_1$ be defined as above. Then $I_1(\kappa) \leq J_1(\kappa)$ on $[0,1]$.*

**Proof.** We observe that the following identities hold for all $\kappa \in [0,1]$ and $\nu \in \left[\varkappa_1, \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right]$:

$$J_1(\kappa) = \int_{\varkappa_1}^{\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}} \left[\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{4\varkappa_1\varkappa_2\kappa + (1-\kappa)\nu(3\varkappa_1+\varkappa_2)}\right) + \chi\left(\frac{4\varkappa_1\varkappa_2\nu}{4\kappa(\varkappa_2\nu + \varkappa_1\nu - \varkappa_1\varkappa_2) + (1-\kappa)(\varkappa_1+3\varkappa_2)}\right)\right]\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2}d\nu \quad (16)$$

and

$$I_1(\kappa) = \int_{\varkappa_1}^{\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}} \left[\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa\varkappa_1\varkappa_2 + (1-\kappa)\nu(\varkappa_1+\varkappa_2)}\right) + \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa((\varkappa_1+\varkappa_2)\nu - \varkappa_1\varkappa_2) + (1-\kappa)(\varkappa_1+\varkappa_2)\nu}\right)\right]\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2}d\nu. \quad (17)$$

Let

$$\nu_1 = \frac{4\varkappa_1\varkappa_2\nu}{4\varkappa_1\varkappa_2\kappa + (1-\kappa)\nu(3\varkappa_1+\varkappa_2)},$$
$$\nu_2 = \frac{4\varkappa_1\varkappa_2\nu}{4\kappa(\varkappa_2\nu + \varkappa_1\nu - \varkappa_1\varkappa_2) + (1-\kappa)\nu(\varkappa_1+3\varkappa_2)},$$
$$\tilde{\omega}_1 = \frac{2\varkappa_1\varkappa_2\nu}{2\kappa\varkappa_1\varkappa_2 + (1-\kappa)\nu(\varkappa_1+\varkappa_2)}$$

and

$$\tilde{\omega}_2 = \frac{2\varkappa_1\varkappa_2\nu}{2\kappa((\varkappa_1+\varkappa_2)\nu - \varkappa_1\varkappa_2) + (1-\kappa)(\varkappa_1+\varkappa_2)\nu}$$

for all $\kappa \in [0,1]$ and $\nu \in \left[\varkappa_1, \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right]$.

Hence, by Lemma 2, the following inequality holds for all $\kappa \in [0,1]$ and $\nu \in \left[\varkappa_1, \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right]$:

$$\begin{aligned}
&\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{4\varkappa_1\varkappa_2\kappa + (1-\kappa)\nu(3\varkappa_1+\varkappa_2)}\right)\\
&\quad+\chi\left(\frac{4\varkappa_1\varkappa_2\nu}{4\kappa(\varkappa_2\nu+\varkappa_1\nu-\varkappa_1\varkappa_2) + (1-\kappa)\nu(\varkappa_1+3\varkappa_2)}\right)\\
&\quad\le \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa\varkappa_1\varkappa_2 + (1-\kappa)\nu(\varkappa_1+\varkappa_2)}\right)\\
&\quad+\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\kappa((\varkappa_1+\varkappa_2)\nu-\varkappa_1\varkappa_2) + (1-\kappa)(\varkappa_1+\varkappa_2)\nu}\right).
\end{aligned} \tag{18}$$

Multiplying both sides by $\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2}$ and then integrating over $\left[\varkappa_1, \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right]$, we get that $I_1(\kappa) \le J_1(\kappa)$ for $\kappa \in [0,1]$. □

**Example 3.** *Let $\chi(\nu) = \nu^{-2}$, $\nu \in [1,2]$, then $\chi$ is harmonically convex. Let $\varphi(\nu) = \left(\frac{1}{\nu} - \frac{3}{4}\right)^2$, $\nu \in [1,2]$. It is clear that $\varphi$ is harmonically symmetric with respect to $\frac{4}{3}$. By using the techniques of integration, we observed the following calculations for $0 \le \kappa \le 1$:*

$$\begin{aligned}
I_1(\kappa) &= \frac{1}{2}\int_1^2 \frac{1}{\nu^2}\left(\frac{1}{\nu}-\frac{3}{4}\right)^2\left[\left(\frac{4\nu}{2\kappa(\nu+1)+3(1-\kappa)\nu}\right)^{-2}\right.\\
&\quad\left.+\left(\frac{4\nu}{\kappa(\nu+2)+3(1-\kappa)\nu}\right)^{-2}\right]d\nu = \frac{2\kappa^2+45}{7680}
\end{aligned}$$

*and*

$$\begin{aligned}
J_1(\kappa) &= \frac{1}{2}\int_1^2 \frac{1}{\nu^2}\left(\frac{1}{\nu}-\frac{3}{4}\right)^2\left[\left(\frac{8\nu}{4\kappa(\nu+1)+7(1-\kappa)\nu}\right)^{-2}\right.\\
&\quad\left.+\left(\frac{8\nu}{2\kappa(\nu+2)+5(1-\kappa)\nu}\right)^{-2}\right]d\nu = \frac{3\kappa^2+185}{30720},
\end{aligned}$$

*The Figure 3 below validates the inequality proved in Theorem 20.*

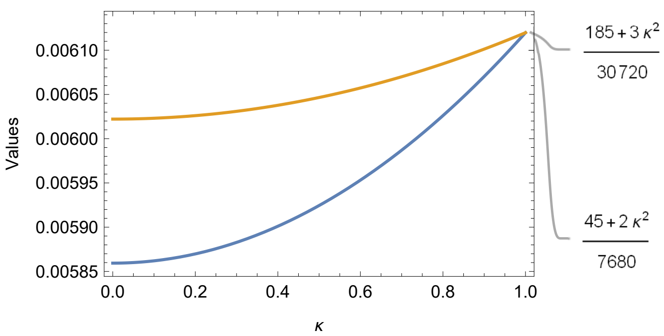


**Figure 3.** The graph of inequality proved in Theorem 20 for $0 \le \kappa \le 1$.

The following result demonstrates how the function attributes of $M_1$ are incorporated:

**Theorem 21.** *Let $\chi, \varphi, M_1$ be defined as above. Then $M_1$ is harmonically convex, increasing on $[0,1]$ and Fejér-type inequality*

$$\begin{aligned}\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{2\varkappa_1\nu}{\varkappa_1+\nu}\right)+\chi\left(\frac{2\varkappa_2\nu}{\nu+\varkappa_2}\right)\right]\frac{\varphi(\nu)}{\nu^2}d\nu = M_1(0)\\ \leq M_1(\kappa)\leq M_1(1)=\frac{1}{2}\left[\chi\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right)+\frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2}\right]\int_{\varkappa_1}^{\varkappa_2}\frac{\varphi(\nu)}{\nu^2}d\nu \quad (19)\end{aligned}$$

*hold for all $\kappa \in [0,1]$.*

**Proof.** We can prove the harmonic convexity of $M_1$ on $[0,1]$ by following the same method as that of proving the harmonic convexity of $M_1$ on $[0,1]$ in Theorem 18.
It is easy to observe that the following identity holds for all $\kappa \in [0,1]$ and $\nu \in \left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right]$:

$$\begin{aligned}M_1(\kappa) = \int_{\varkappa_1}^{\frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}}\Bigg[\chi\left(\frac{\varkappa_2\nu}{\kappa\nu+(1-\kappa)\varkappa_2}\right)+\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{(3\varkappa_1+\varkappa_2)\nu-2(1-\kappa)\varkappa_1\varkappa_2-2\kappa\varkappa_1\nu}\right)\\ +\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2(1-\kappa)\varkappa_1\varkappa_2+(\varkappa_2-\varkappa_1)\nu+2\kappa\varkappa_1\nu}\right)\\ +\chi\left(\frac{\varkappa_1\varkappa_2\nu}{(\varkappa_1+\varkappa_2-\kappa\varkappa_1)\nu-(1-\kappa)\varkappa_1\varkappa_2}\right)\Bigg]\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2}d\nu.\end{aligned}$$

According to Lemma 2, the following inequalities are valid for all $\kappa_1, \kappa_2 \in [0,1]$ with $\kappa_1 < \kappa_2$ and $\nu \in \left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right]$:

$$\begin{aligned}\chi\left(\frac{\varkappa_2\nu}{\kappa_1\nu+(1-\kappa_1)\varkappa_2}\right)+\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{(3\varkappa_1+\varkappa_2)\nu-2(1-\kappa_1)\varkappa_1\varkappa_2-2\kappa_1\varkappa_1\nu}\right)\\ \leq \chi\left(\frac{\varkappa_2\nu}{\kappa_2\nu+(1-\kappa_2)\varkappa_2}\right)+\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{(3\varkappa_1+\varkappa_2)\nu-2(1-\kappa_2)\varkappa_1\varkappa_2-2\kappa_2\varkappa_1\nu}\right) \quad (20)\end{aligned}$$

$$\begin{aligned}\chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2(1-\kappa_1)\varkappa_1\varkappa_2+(\varkappa_2-\varkappa_1)\nu+2\kappa_1\varkappa_1\nu}\right)+\chi\left(\frac{\varkappa_1\varkappa_2\nu}{(\varkappa_1+\varkappa_2-\kappa_1\varkappa_1)\nu-(1-\kappa_1)\varkappa_1\varkappa_2}\right)\\ \leq \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2(1-\kappa_2)\varkappa_1\varkappa_2+(\varkappa_2-\varkappa_1)\nu+2\kappa_2\varkappa_1\nu}\right)\\ +\chi\left(\frac{\varkappa_1\varkappa_2\nu}{(\varkappa_1+\varkappa_2-\kappa_2\varkappa_1)\nu-(1-\kappa_2)\varkappa_1\varkappa_2}\right). \quad (21)\end{aligned}$$

Adding (20) and (21) and multiplying both sides of the resulting inequality by $\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2}$ and then integrating over $\left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right]$, we get that $M_1(\kappa_1) \leq M_1(\kappa_2)$ for $\kappa_1, \kappa_2 \in [0,1]$ with $\kappa_1 < \kappa_2$. Hence, $M_1$ is increasing on $[0,1]$ and thus the inequalities (19) follow. $\square$

**Example 4.** *Let $\chi(\nu) = \nu^{-2}$, $\nu \in [1,2]$, then $\chi$ is harmonically convex. Let $\varphi(\nu) = \left(\frac{1}{\nu}-\frac{3}{4}\right)^2$, $\nu \in [1,2]$. It is clear that $\varphi$ is harmonically symmetric with respect to $\frac{4}{3}$. By using the techniques of integration, we observed the following calculations for $0 \leq \kappa \leq 1$:*

$$M_1(0) = \frac{1}{2}\left[\left(\frac{8}{7}\right)^{-2}+\left(\frac{8}{5}\right)^{-2}\right]\int_1^2\frac{1}{\nu^2}\left(\frac{1}{\nu}-\frac{3}{4}\right)^2 d\nu = \frac{37}{6144},$$

$$M_1(\kappa) = \frac{1}{2}\int_1^{\frac{4}{3}} \left[ \left( \frac{4\nu}{2\nu\kappa + (1-\kappa)(\nu+2)} \right)^{-2} + \left( \frac{4\nu}{3\kappa\nu + 2(1-\kappa)(1+\nu)} \right)^{-2} \right] \frac{\left(\frac{1}{\nu} - \frac{3}{4}\right)^2}{\nu^2} d\nu$$
$$+ \frac{1}{2}\int_{\frac{4}{3}}^{2} \left[ \left( \frac{4\nu}{3\kappa\nu + (1-\kappa)(2+\nu)} \right)^{-2} + \left( \frac{2\nu}{2\nu\kappa + (1-\kappa)(\nu+1)} \right)^{-2} \right] \frac{\left(\frac{1}{\nu} - \frac{3}{4}\right)^2}{\nu^2} d\nu$$
$$= \frac{376 - 27\kappa + 31\kappa^2}{61440}$$

*and*

$$M_1(1) = \frac{1}{2}\left[ \left(\frac{4}{3}\right)^{-2} + \frac{1^{-2} + 2^{-2}}{2} \right] \int_1^2 \frac{1}{\nu^2}\left(\frac{1}{\nu} - \frac{3}{4}\right)^2 d\nu = \frac{47}{7680}.$$

*The Figure 4 below validates the inequality (19) in Theorem 21.*

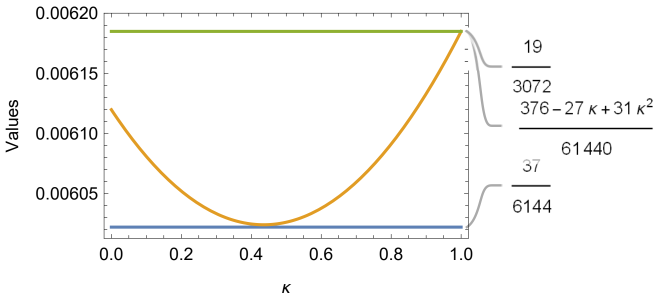


**Figure 4.** The graph of inequality (19) for $0 \leq \kappa \leq 1$.

The properties of the mapping $N_1$ are presented in the given result:

**Theorem 22.** *Let $\chi, \varphi, N_1$ be defined as above. Then $N_1$ is harmonically convex, increasing on $[0,1]$ and the Fejér-type inequalities*

$$\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2} \left[ \chi\left( \frac{2\varkappa_1\nu}{\varkappa_1 + \nu} \right) + \chi\left( \frac{2\varkappa_2\nu}{\nu + \varkappa_2} \right) \right] \frac{\varphi(\nu)}{\nu^2} d\nu = N_1(0)$$
$$\leq N_1(\kappa) \leq N_1(1) = \frac{\chi(\varkappa_1) + \chi(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} \frac{\varphi(\nu)}{\nu^2} d\nu \quad (22)$$

*hold for all $\kappa \in [0,1]$.*

**Proof.** We can prove the harmonic convexity of $N_1$ on $[0,1]$ by following the same method as that of proving the harmonic convexity of $I_1$ on $[0,1]$ in Theorem 18.
It is easy to observe that

$$N_1(\kappa) = \int_{\varkappa_1}^{\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}} \left[ \chi\left( \frac{\varkappa_1\varkappa_2\nu}{\varkappa_1\nu\kappa + (1-\kappa)(\varkappa_1\nu + \varkappa_2\nu - \varkappa_1\varkappa_2)} \right) \right.$$
$$\left. + \chi\left( \frac{\varkappa_1\nu}{\kappa\nu + (1-\kappa)\varkappa_1} \right) \right] \frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1 - \nu}\right)}{\nu^2} d\nu.$$

holds for all $\kappa \in [0,1]$ and $\nu \in \left[\varkappa_1, \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right]$.
According to Lemma 2, the given inequalities are valid for all $\kappa_1, \kappa_2 \in [0,1]$ with $\kappa_1 < \kappa_2$ and $\nu \in \left[\varkappa_1, \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right]$:

$$\begin{aligned}\chi\left(\frac{\varkappa_1\nu}{\kappa_1\nu+(1-\kappa_1)\varkappa_1}\right)+\chi\left(\frac{\varkappa_1\varkappa_2\nu}{\varkappa_1\nu\kappa_1+(1-\kappa_1)(\varkappa_1\nu+\varkappa_2\nu-\varkappa_1\varkappa_2)}\right)\\ \leq \chi\left(\frac{\varkappa_1\nu}{\kappa_2\nu+(1-\kappa_2)\varkappa_1}\right)+\chi\left(\frac{\varkappa_1\varkappa_2\nu}{\varkappa_1\nu\kappa_2+(1-\kappa_2)(\varkappa_1\nu+\varkappa_2\nu-\varkappa_1\varkappa_2)}\right).\end{aligned} \tag{23}$$

Multiplying both sides of (23) by $\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2}$ and then integrating over $\left[\varkappa_1, \frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right]$, we get that $N_1(\kappa_1) \leq N_1(\kappa_2)$ for $\kappa_1, \kappa_2 \in [0,1]$ with $\hat{\kappa}_1 < \kappa_2$. Hence, $N_1$ is increasing on $[0,1]$ and thus the inequalities (22) follow. $\square$

**Example 5.** *Let $\chi(\nu) = \nu^{-2}$, $\nu \in [1,2]$, then $\chi$ is harmonically convex. Let $\varphi(\nu) = \left(\frac{1}{\nu} - \frac{3}{4}\right)^2$, $\nu \in [1,2]$. It is clear that $\varphi$ is harmonically symmetric with respect to $\frac{4}{3}$. By using the techniques of integration, we observed the following calculations for $0 \leq \kappa \leq 1$:*

$$N_1(0) = \frac{1}{2}\left[\left(\frac{8}{7}\right)^{-2}+\left(\frac{8}{5}\right)^{-2}\right]\int_1^2 \frac{1}{\nu^2}\left(\frac{1}{\nu}-\frac{3}{4}\right)^2 d\nu = \frac{37}{6144},$$

$$\begin{aligned}N_1(\kappa) = \frac{1}{2}\int_1^2 \Bigg[\left(\frac{4\nu}{2\nu\kappa+(1-\kappa)(\nu+2)}\right)^{-2}\\ +\left(\frac{2\nu}{2\nu\kappa+(1-\kappa)(1+\nu)}\right)^{-2}\Bigg]\frac{\left(\frac{1}{\nu}-\frac{3}{4}\right)^2}{\nu^2}d\nu = \frac{2\kappa^2+\kappa+47}{7680}\end{aligned}$$

*and*

$$N_1(1) = \frac{1^{-2}+2^{-2}}{2}\int_1^2 \frac{1}{\nu^2}\left(\frac{1}{\nu}-\frac{3}{4}\right)^2 d\nu = \frac{5}{768}.$$

*The Figure 5 below validates the inequality proved in Theorem 22.*

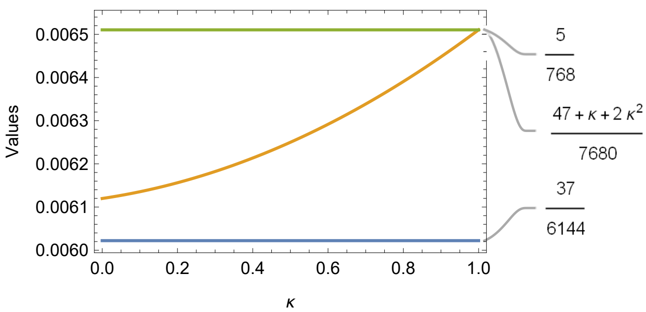


**Figure 5.** The graph of inequality proved in Theorem 20 for $0 \leq \kappa \leq 1$.

**Remark 2.** *Let $\varphi(\nu) = \frac{\varkappa_1\varkappa_2}{\varkappa_2-\varkappa_1}$, $\nu \in [\varkappa_1, \varkappa_2]$ in Theorem 18. Then $N_1(\kappa) = V(\kappa)$, $\kappa \in [0,1]$ and the inequalities (22) become*

$$\frac{\varkappa_1\varkappa_2}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \frac{\chi(\nu)}{\nu^2}d\nu = V(0) \leq V(\kappa) \leq V(1) = \frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2},$$

*where $V$ is defined by (7).*

**Theorem 23.** *Let $\chi$, $\varphi$, $M_1$, $N_1$ be defined as above. Then $M_1(\kappa) \leq N_1(\kappa)$ on $[0,1]$.*

**Proof.** We observe that the following identities hold for all $\kappa \in [0,1]$ and $\nu \in \left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right]$:

$$\begin{aligned} M_1(\kappa) = \int_{\varkappa_1}^{\frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}} \Bigg[ & \chi\left(\frac{\varkappa_2\nu}{\kappa\nu+(1-\kappa)\varkappa_2}\right) + \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{(3\varkappa_1+\varkappa_2)\nu - 2(1-\kappa)\varkappa_1\varkappa_2 - 2\kappa\varkappa_1\nu}\right) \\ & + \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2(1-\kappa)\varkappa_1\varkappa_2 + (\varkappa_2-\varkappa_1)\nu + 2\kappa\varkappa_1\nu}\right) \\ & + \chi\left(\frac{\varkappa_1\varkappa_2\nu}{(\varkappa_1+\varkappa_2-\kappa\varkappa_1)\nu - (1-\kappa)\varkappa_1\varkappa_2}\right)\Bigg] \frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2} d\nu. \end{aligned} \tag{24}$$

and

$$\begin{aligned} N_1(\kappa) = \int_{\varkappa_1}^{\frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}} \Bigg[ & \chi\left(\frac{\varkappa_2\nu}{\kappa\nu+(1-\kappa)\varkappa_2}\right) + \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\varkappa_1\nu\kappa + (1-\kappa)((3\varkappa_1+\varkappa_2)\nu - 2\varkappa_1\varkappa_2)}\right) \\ & + \chi\left(\frac{\varkappa_1\varkappa_2\nu}{\kappa\varkappa_2\nu + (1-\kappa)(\varkappa_2\nu + \varkappa_1\nu - \varkappa_1\varkappa_2)}\right) \\ & + \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\varkappa_2\nu\kappa + (1-\kappa)(2\varkappa_1\varkappa_2 + (\varkappa_2-\varkappa_1)\nu)}\right)\Bigg] \frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1-\nu}\right)}{\nu^2} d\nu. \end{aligned} \tag{25}$$

Let

$$\begin{aligned} \nu_1 &= \frac{\varkappa_2\nu}{\kappa\nu + (1-\hat{\kappa})\varkappa_2}, \\ \nu_2 &= \frac{2\varkappa_1\varkappa_2\nu}{(3\varkappa_1+\varkappa_2)\nu - 2(1-\kappa)\varkappa_1\varkappa_2 - 2\kappa\varkappa_1\nu}, \\ \tilde{\omega}_1 &= \frac{\varkappa_2\nu}{\kappa\nu+(1-\kappa)\varkappa_2} \end{aligned}$$

and

$$\tilde{\omega}_2 = \frac{2\varkappa_1\varkappa_2\nu}{2\varkappa_1\nu\kappa + (1-\kappa)((3\varkappa_1+\varkappa_2)\nu - 2\varkappa_1\varkappa_2)}$$

for all $\kappa \in [0,1]$ and $\nu \in \left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right]$.
Hence, Lemma 2 gives the inequality

$$\begin{aligned} & \chi\left(\frac{\varkappa_2\nu}{\kappa\nu+(1-\kappa)\varkappa_2}\right) + \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{(3\varkappa_1+\varkappa_2)\nu - 2(1-\kappa)\varkappa_1\varkappa_2 - 2\kappa\varkappa_1\nu}\right) \\ & \quad \leq \chi\left(\frac{\varkappa_2\nu}{\kappa\nu+(1-\kappa)\varkappa_2}\right) + \hat{\chi}\left(\frac{2\varkappa_1\varkappa_2\nu}{2\varkappa_1\nu\kappa + (1-\kappa)((3\varkappa_1+\varkappa_2)\nu - 2\varkappa_1\varkappa_2)}\right) \end{aligned} \tag{26}$$

for all $\kappa \in [0,1]$ and $\nu \in \left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right]$.
Similarly with the choices

$$\begin{aligned} \nu_1 &= \frac{2\varkappa_1\varkappa_2\nu}{2(1-\kappa)\varkappa_1\varkappa_2 + (\varkappa_2-\varkappa_1)\nu + 2\kappa\varkappa_1\nu}, \\ \nu_2 &= \frac{\varkappa_1\varkappa_2\nu}{(\varkappa_1+\varkappa_2-\kappa\varkappa_1)\nu - (1-\kappa)\varkappa_1\varkappa_2}, \\ \tilde{\omega}_1 &= \frac{\varkappa_1\varkappa_2\nu}{\kappa\varkappa_2\nu + (1-\kappa)(\varkappa_2\nu + \varkappa_1\nu - \varkappa_1\varkappa_2)}, \end{aligned}$$

$$\tilde{\omega}_2 = \frac{2\varkappa_1\varkappa_2\nu}{2\varkappa_2\nu\kappa + (1-\kappa)(2\varkappa_1\varkappa_2 + (\varkappa_2 - \varkappa_1)\nu)}$$

in Lemma 2, we get

$$\chi\left(\frac{\varkappa_1\varkappa_2\nu}{\kappa\varkappa_2\nu + (1-\kappa)(\varkappa_2\nu + \varkappa_1\nu - \varkappa_1\varkappa_2)}\right) + \chi\left(\frac{\varkappa_1\varkappa_2\nu}{(\varkappa_1 + \varkappa_2 - \kappa\varkappa_1)\nu - (1-\kappa)\varkappa_1\varkappa_2}\right)$$
$$\leq \chi\left(\frac{\varkappa_1\varkappa_2\nu}{\kappa\varkappa_2\nu + (1-\kappa)(\varkappa_2\nu + \varkappa_1\nu - \varkappa_1\varkappa_2)}\right)$$
$$+ \chi\left(\frac{2\varkappa_1\varkappa_2\nu}{2\varkappa_2\nu\kappa + (1-\kappa)(2\varkappa_1\varkappa_2 + (\varkappa_2 - \varkappa_1)\nu)}\right). \quad (27)$$

Adding (26) and (27) and multiplying the result by $\frac{\varphi\left(\frac{\varkappa_1\nu}{2\varkappa_1 - \nu}\right)}{\nu^2}$ and then integrating over $\left[\varkappa_1, \frac{4\varkappa_1\varkappa_2}{3\varkappa_1 + \varkappa_2}\right]$, we get that $M_1(\kappa) \leq N_1(\kappa)$ for $\kappa \in [0,1]$. $\square$

**Example 6.** *Let $\chi(\nu) = \nu^{-2}$, $\nu \in [1,2]$, then $\chi$ is harmonically convex. Let $\varphi(\nu) = \left(\frac{1}{\nu} - \frac{3}{4}\right)^2$, $\nu \in [1,2]$. It is clear that $\varphi$ is harmonically symmetric with respect to $\frac{4}{3}$. By using the techniques of integration, we observed the following calculations for $0 \leq \kappa \leq 1$:*

$$M_1(\kappa) = \frac{1}{2}\int_1^{\frac{4}{3}}\left[\left(\frac{4\nu}{2\nu\kappa + (1-\kappa)(\nu+2)}\right)^{-2} + \left(\frac{4\nu}{3\kappa\nu + 2(1-\kappa)(1+\nu)}\right)^{-2}\right]\frac{\left(\frac{1}{\nu} - \frac{3}{4}\right)^2}{\nu^2}d\nu$$
$$+ \frac{1}{2}\int_{\frac{4}{3}}^{2}\left[\left(\frac{4\nu}{3\kappa\nu + (1-\kappa)(2+\nu)}\right)^{-2} + \left(\frac{2\nu}{2\nu\kappa + (1-\kappa)(\nu+1)}\right)^{-2}\right]\frac{\left(\frac{1}{\nu} - \frac{3}{4}\right)^2}{\nu^2}d\nu$$
$$= \frac{31\kappa^2 - 27\kappa + 376}{61440}$$

*and*

$$N_1(\kappa) = \frac{1}{2}\int_1^2\left[\left(\frac{4\nu}{2\nu\kappa + (1-\kappa)(\nu+2)}\right)^{-2}\right.$$
$$\left. + \left(\frac{2\nu}{2\nu\kappa + (1-\kappa)(1+\nu)}\right)^{-2}\right]\frac{\left(\frac{1}{\nu} - \frac{3}{4}\right)^2}{\nu^2}d\nu = \frac{2\kappa^2 + \kappa + 47}{7680}$$

*The Figure 6 below validates the inequality proved in Theorem 23.*

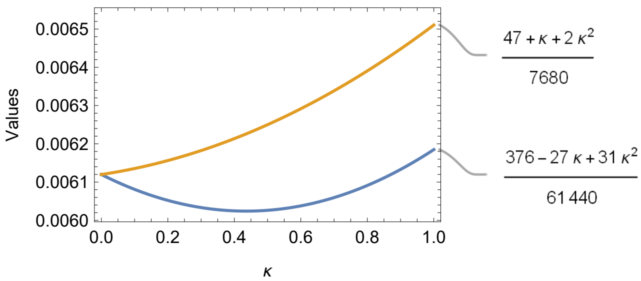


**Figure 6.** The graph of inequality proved in Theorem 23 for $0 \leq \kappa \leq 1$.

Theorems 19–23 lead to the following Fejér-type inequalities.

**Corollary 1.** *Let $\chi$, $\varphi$ be defined as above. Then*

$$\begin{aligned}\chi\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\varphi(\nu)}{\nu^2}d\nu &\le \frac{\chi\left(\frac{4\varkappa_1\varkappa_2}{\varkappa_1+3\varkappa_2}\right)+\chi\left(\frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{\varphi(\nu)}{\nu^2}d\nu\\ &\le \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\chi\left(\frac{2\varkappa_1\nu}{\varkappa_1+\nu}\right)+\chi\left(\frac{2\varkappa_2\nu}{\nu+\varkappa_2}\right)\right]\frac{\varphi(\nu)}{\nu^2}d\nu\\ &\le \frac{1}{2}\left[\chi\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right)+\frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2}\right]\int_{\varkappa_1}^{\varkappa_2}\frac{\varphi(\nu)}{\nu^2}d\nu\\ &\le \frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{\varphi(\nu)}{\nu^2}d\nu. \end{aligned} \tag{28}$$

**Corollary 2.** *Let $\varphi(\nu)=\frac{\varkappa_1\varkappa_2}{\varkappa_2-\varkappa_1}$, $\nu\in[\varkappa_1,\varkappa_2]$ in Corollary 1. Then the inequality (28) reduces to*

$$\begin{aligned}\chi\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right) &\le \frac{\chi\left(\frac{4\varkappa_1\varkappa_2}{\varkappa_1+3\varkappa_2}\right)+\chi\left(\frac{4\varkappa_1\varkappa_2}{3\varkappa_1+\varkappa_2}\right)}{2}\\ &\le \frac{1}{2}\left(\frac{\varkappa_1\varkappa_2}{\varkappa_2-\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{1}{\nu^2}\left[\chi\left(\frac{2\varkappa_1\nu}{\varkappa_1+\nu}\right)+\chi\left(\frac{2\varkappa_2\nu}{\nu+\varkappa_2}\right)\right]d\nu\\ &\le \frac{1}{2}\left[\chi\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right)+\frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2}\right]\le\frac{\chi(\varkappa_1)+\chi(\varkappa_2)}{2}. \end{aligned} \tag{29}$$

## 3. Conclusions

Integral inequalities have become an emerging topic in the last three decades. Researchers are trying to find new proofs of the existing results and trying to investigate refinements of the existing results using novel ideas. In different directions of research in the field of inequalities and other fields of mathematics, convexity plays an important role in establishing new results and refinements of the existing results. Mathematicians are trying to find new and novel generalizations to generalize and refine the existing results. One the generalizations of the convex functions is known as harmonically convex functions, which has given rise to a number of novel results and refinements. In this study, we defined new mappings over the interval $[0,1]$ related to the Hermite-Hadamard and Fejér type inequalities, proved for harmonically-convex functions and obtained new Hermite-Hadamard and Fejér type inequalities using novel techniques and notions of harmonically convexity. The results obtained are not only the new Fejér type inequalities but also provide refinements of Hermite-Hadamard and Fejér type results already proven in the existing literature of mathematical inequalities. The research of this paper could be a source of inspiration for new researchers and for the researchers already working in the field of mathematical inequalities.

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