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Attached Flows for Reaction–Diffusion Processes Described by a Generalized Dodd–Bullough–Mikhailov Equation

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Abstract: This paper uses the attached flow method for solving nonlinear second-order differential equations of the reaction–diffusion type. The key steps of the method consist of the following: (i) reducing the differentiability order by defining the first derivative of the variable as a new variable called the flow and (ii) a forced decomposition of the derivative-free term so that the flow appears explicitly in it. The resulting reduced equation is solved using specific balancing rules. Only step (i) would lead to an Abel-type equation with complicated integral solutions. Completed with (ii) and with the graduation procedure, the attached flow method used in the paper, without requiring such a great effort, allows for the obtaining of accurate analytical solutions. The method is applied here to a subclass of reaction–diffusion equations, the generalized Dodd–Bulough–Mikhailov equation, which includes a translation of the variable and nonlinearities up to order five. The equation is solved for each order of nonlinearity, and the solutions are discussed following the values of the parameters involved in the equation.

Keywords: attached flow method; reaction–diffusion equations; generalized Dodd–Bulough–Mikhailov equation

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1. Introduction

In a previous paper [1], we introduced an improved approach to the rather classical method for solving nonlinear second-order differential equations, based on the use of the first derivative as a new variable and on the investigation of the resulting first-order equation in this new variable [2]. Despite the apparent simplification generated via the reduction of the differentiability order, the method is not easy to apply, as it leads to an Abel-type equation [3]. Precisely to overcome this situation, the approach we proposed, named the attached flow method, contains an additional step: a forced decomposition of the term in the equation that does not contain derivatives. To be very specific, we consider a rather wide class of second-order equations, the reaction–diffusion equation:

$$A(u)\, u'' + B(u)\, u'^2 + C(u)\, u' + E(u) = 0. \tag{1}$$

Equation (1) is the explicit form of the ordinary differential equation generated after applying the wave transformation $\xi = x - Vt$ to the general nonlinear diffusion–convection–reaction equation with functional coefficients. In the $2D$ space defined by the coordinates $\{x, t\}$, such an equation can be written as follows:

$$u_t = (A(u)u_x)_x + [C(u) - V]u_x + E(u). \tag{2}$$

It is a general equation describing nonlinear phenomena from various fields, such as nonlinear optics [4], plasma physics [5,6], or heat and fluids diffusion [7]. From mathematical point of view, there are many interesting approaches related to the stability and convergence analysis of this 2D equation, to its variational structure, or to a positivity-preserving and energy-stable operator splitting scheme [8,9].

As we are now interested in the traveling waves solutions, we will consider here exclusively the 1D Equation (1).

In [1], we imposed $A(u)$, $B(u)$, and $C(u)$ to be monomials, with $A(u) \neq 0$, while $E(u)$ had the freedom to be a polynomial with minimum and maximum degrees in u denoted as $n(E)$, respectively $N(E)$. How the method works was illustrated for the following cases: (i) $A(u) \neq 0$, $B(u) = C(u) = 0$, $E(u) \neq 0$; (ii) $A(u) \neq 0$, $B(u) = 0$, $C(u) \neq 0$, and $E(u) \neq 0$; (iii) none of the coefficients are vanishing, but $E(u)$ becomes a monomial. The case when $C(u)$ is the only vanishing coefficient was only announced as requiring special precautions, with the mention that it would be addressed later. The approach of this case represents the main objective of the current paper.

The paper is structured as follows: after these introductory notes, the attached flow method is revised in the second section. The third section is devoted to the presentation of a specific case of (1) with $C(u) = 0$, a new version of the generalized Dodd–Bulough–Mikhailov (gDBM) equation that, to avoid the solving difficulties, includes a simple translation of the variable. How the attached flow method has to be applied to this model will be investigated in the fourth section of the paper, while the last section will present a synthesis of the reported results with some supplementary comments on the method. We do not claim to report new DBM solutions here; the real objective of the work is to see, using this new gDBM as a toy model, how the general algorithm of the attached flow method should be applied to equations in this class.

2. Basic Facts on the Attached Flow Method

Let us now summarize the key aspects presented in [1] for the attached flow method. Starting from a general nonlinear second-order differential equation of reaction–diffusion type [10], the method supposes two reductions: (i) the reduction in the differentiability order by defining the first derivative of the variable [11,12] as a new variable called the flow, and (ii) the reduction in the nonlinearity degree by decomposing the derivative-free term so that the flow appears explicitly in it [1]. The reduction in the nonlinearity avoids reaching an Abel-type equation, which is difficult to solve analytically [11]. The previous two reductions lead to a simpler equation that, in our approach, is easily solved using specific balancing rules. This supplementary balancing makes the attached flow method very effective in looking for analytic solutions that, in other previous approaches, appear in complicated integral forms and need full effort to be obtained.

Explicitly, the method is based on three main requirements:

- The variable $u(\xi)$ must supplementarily satisfy a flow equation of the following form:

$$u'(\xi) = f(u). \tag{3}$$

Relation (3) represents the classic requirement for reducing the differentiability order recommended in textbooks. It transforms (1) in a particular case of the second kind of Abel equation in the variable $f(u)$:

$$A(u)f(u)\frac{df}{du} + B(u)f(u)^2 + C(u)f(u) + E(u) = 0, \tag{4}$$

- The flow $f(u)$ must be connected with $E(u)$ via a forced decomposition of the following form:

$$E(u) = f(u) \cdot h(u). \tag{5}$$

This is the novelty proposed by us, and it aims to avoid the direct solving of Equation (4) that usually only has implicit solutions. The relation (5) transforms (1) into the following first-order differential equation:

$$A(u)\frac{df}{du} + B(u)\,f(u) + C(u) + h(u) = 0, \tag{6}$$

- The two functions from (5), $f(u)$ and $h(u)$, are considered polynomials of the following form:

$$f(u) = \sum_{i=n(f)}^{N(f)} \alpha_i u^i;\; h(u) = \sum_{j=n(h)}^{N(h)} \beta_j u^j. \tag{7}$$

The summation limits in (7) are established through a balancing procedure, imposing the requirement to compensate the different powers of u in relations (5) and (6).

The steps in solving (1) now become the following: (i) to determine the flow $f(u)$ and (ii) to find the solution $u(\xi)$ by integrating (3). The first step is supposed to consider (7) in Equations (5) and (6) and to determine the possible form of $f(u)$ via an adequate balancing procedure.

Two important clarifications need to be made at this point:

- It is clear that the decomposition (5) is not unique. Various solutions can be generated, supposing $f(u)$ and $h(u)$ as polynomials with degrees running from unit to unit, between $\{n(f)$ and $N(f)\}$, respectively $\{n(h)$ and $N(h)\}$. However, a single decomposition proves to be compatible with solving a certain nonlinear model. It is established in the attached flow approach through a specific balancing process, another element of originality of the method.
- In general $\{\alpha_i, \beta_j\}$ are constant coefficients. In specific cases that will be mentioned below, we will be obliged to consider them as functions of the independent variable:

$$\alpha_i = \alpha_i(\xi); \beta_j = \beta_j(\xi).$$

In these cases, the flow Equation (3) will become the following:

$$u'(\xi) = f(\xi, u(\xi)) = \sum_{i=n(f)}^{N(f)} \alpha_i(\xi) u^i(\xi). \tag{8}$$

With (8), the reduced Equation (6) becomes the following:

$$A(u)\left[\frac{\partial f}{\partial \xi} + \frac{df}{du} f\right] + B(u)\,f(u) + C(u) + h(u) = 0. \tag{9}$$

With these remarks in mind, let us now return to the problem of determining the flow, $f(u)$. This is now equivalent to determining the coefficients α_i, and $N(f)$, $n(f)$, the maximum, and, respectively, the minimum degrees in u from (7) or (8). The relation (5) imposes, as necessary but not sufficient conditions, the following requirements:

$$N(f) + N(h) = N(E); n(f) + n(h) = n(E). \tag{10}$$

The requirement for mutual compensation of the terms with different powers in u that appear in (6) leads to an additional set of algebraic relations that, in principle, suggests the admissible forms of the flow $f(u)$ or more precisely leads to the following conclusions on the maximal degrees in u of the flow $f(u)$ [1]:

$$\begin{aligned}
N(f) &= N(C) - N(A) + 1 \text{ if } N(B) \le N(A) - 1 \le 2N(C) - N(E);\\
N(f) &= \frac{N(E) - N(A) + 1}{2} \text{ if } \left\{ \begin{array}{c} N(B) \le N(A) - 1 \\ 2N(C) - N(E) \le N(A) - 1 \end{array} \right. ;\\
N(f) &= N(C) - N(B) \text{ if } N(A) - 1 \le N(B) \le 2N(C) - N(E);\\
N(f) &= \frac{N(E) - N(B)}{2} \text{ if } \left\{ \begin{array}{c} N(A) - 1 \le N(B) \\ 2N(C) - N(E) \le N(B) \end{array} \right. ;
\end{aligned} \tag{11}$$

$$N(f) \quad = \quad N(E) - N(C) \text{ if } \begin{cases} N(A) - 1 \le 2N(C) - N(E) \\ N(B) \ge N(E) - 2N(C) \end{cases} .$$

These degrees are not compulsory integers. How exactly the mechanism is functioning will be illustrated in the fourth section of the paper with a specific sub-equation of (1) with $C(u) = 0$, namely the gDBM equation.

3. The DBM Model as a Reaction–Diffusion Equation

As announced, the aim of this article is to show how the attached flow as a solving method can be applied on reaction–diffusion equations of the form (1) with $C(u) = 0$. To be very specific, we will illustrate this using the DBM model, important in many problems from nonlinear optics, hydrodynamics, or quantum field theory. Despite numerous studies on the DBM equation and the various techniques used for solving it, the model is surprisingly rich, and depending on the number of parameters each approach introduces in the study, new classes of solutions can be generated. An important note has to be made here: there are maybe dozens of recent papers claiming that new DBM solutions have been obtained, but in many cases, these apparently "new" solutions are nothing more than solutions expressed as slightly different mathematical expressions belonging to already known classes of solutions. This is one of the current errors pointed out in [13], and this is why, to avoid misunderstandings, we have already stated that we will not introduce new solutions in this paper, but will only illustrate how the attached flow method can be used in the case of the DBM equation. A possible classification of the DBM solutions is suggested in [14].

In the $2D$ space of the coordinates $\{x, t\}$, the DBM model is described by a function, $v(x, t)$, which satisfies the following equation:

$$w_{xt} + e^{w} + e^{-2w} = 0. \tag{12}$$

At the classical level, (12) is known as a nonlinear integrable model describing both classical and quantum propagation phenomena. In the quantum context, the equation appears as the Bullough–Dodd model for scalar fields [15], and it is equivalent to an affine Toda field with zero curvature representation in twisted affine Kac–Moody algebras [16]. This feature assures the existence of soliton-like solutions.

Equation (12) can be included, in turn, in a more general equation, the generalized Tzitzeica–Dodd–Bullough–Mikhailov (gTDBM) equation [17]:

$$w_{xt} + pe^{\sigma w} + qe^{\rho w} = 0. \tag{13}$$

From this form, we can recover not only the simple DBM Equation (12) but also other interesting models, such as, for example, the *sinh*- or *cosh*–Gordon equations [18] and the Liouville equation, corresponding to $q = 0$.

Due to the non-polynomial form of (13), special techniques have to be applied in order to determine its solutions. The classical approach consists of using the transformation $w(x, t) = \ln \phi(x, t)$ with the master Equation (12) taking the following form:

$$\phi_{xt}\phi - \phi_x\phi_t + p\phi^{\sigma+2} + q\phi^{\rho+2} = 0. \tag{14}$$

Equation (14) contains four parameters: the real parameters p, q, respectively, the integers σ, ρ. This number can be reduced using suitable choices. We will discuss here various reduced equations with fewer parameters. The starting point will be a slightly different form of the previous equation, usually called the generalized DBM equation [14]. It corresponds to the following choices: $\sigma + 2 = m$, $\rho = -2$, choices that lead to the following:

$$\phi_{xt}\phi - \phi_x\phi_t + p\phi^{m} + q = 0. \tag{15}$$

This will be the main equation considered in this paper, to which we will apply the attached flow. The method allows the traveling wave solutions to be found, and it supposes, as a first step to transform (15) into a nonlinear ordinary differential equation (NODE) by taking into consideration the wave transformation, $\xi = x - Vt$. With it and using the notations $\phi(x,t) \to u(\xi)$, $u' = du/d\xi$, $u'' = d^2u/d\xi^2$, Equation (15) can be written as follows:

$$-Vuu'' + Vu'^2 + pu^m + q = 0. \tag{16}$$

A first remark is that the DBM model (12) corresponds to (16) with $p = 1, q = 1$, and $m = 3$:

$$-Vuu'' + Vu'^2 + u^3 + 1 = 0. \tag{17}$$

A second remark is that (16) can be generalized by including a translation of the dependent variable:

$$u \to v \equiv u + k,\ k = const. \tag{18}$$

The main equation to be studied becomes the following:

$$-V(u+k)u'' + Vu'^2 + p(u+k)^m + q = 0, \tag{19}$$

where m is the nonlinearity order, p and q are the constant coefficients of the free derivative term, V is the wave velocity, and k is the translation constant.

Equation (19) is what we will understand here by the generalized Dodd–Bullough–Mikhailov equation. In its version with $k = 0$, it was intensively studied before; see, for example, [17]. This version includes what is called the DBM equation as the particular case, $m = 3$. The translation with $k \neq 0$ is what our paper proposes as a novelty, and as we will see, it is essential to finding the minimal degree n(f) for the flow (7).

By translating the variable, the supplementary term $-Vku''$ breaks the initial symmetry between the minimal degrees of $f(u)$ and $h(u)$, making possible $n(f) \neq n(h)$, a key fact in solving the problem with our method. The derivative-free term generated in the same process now contains a full spectrum of powers in u and those terms, together with derivative ones, allow the solving of the reduced equation: compatible expressions for the two functions may be generated now, when the specific balancing of the attached flow approach is applied. We will discuss in the next section how this can be done and what the results are when nonlinearities given via $m = \{1,2,3,4,5\}$ are considered.

Let us mention that the DBM equation has been solved so far with various other methods. The simplest approaches tend to use a predefined form of solutions, as in [19] or [20], where the tanh method is applied. More generally, traveling wave solutions for DBM have been generated via different methods belonging to the auxiliary equation technique: the $\left(\frac{G'}{G}\right)$-expansion [21], the functional expansion [22], or the sub-equation technique, considering the elliptic Jacobi as an auxiliary equation [15]. Soliton and compacton-like solutions were generated in [23] with the exponential function method, or in [24]with the generalized Kudryashov method and an improved F-expansion method. Other investigations were based on the truncated Painlevé expansion [25], Darboux transformation [26], or integral bifurcation [17]. Analytic solutions for DBM were reported in [27] using Lax operators, an approach that, together with Hirota's method, provides important tools for investigating the complete integrability of nonlinear models [28,29]. An invariant group of solutions could also be obtained using the symmetry reduction method [30]. In [31] the DBM equation was approached based on the first integral method, the closest approach to the attached flow that we will use here.

4. Solving DBM with Attached Flows

Let us consider now how the gDBM Equation (19) could be solved directly using the attached flow method. As it has the form of the generalized reaction–diffusion Equation (1), we can transfer the results mentioned for this last equation in Section 2 from above. It is simple to see that the following applies:

$$A(u) = -V(u+k); B(u) = V; C(u) = 0; E(u) = p(u+k)^m + q. \tag{20}$$

So, $A(u)$ is a polynomial with $N(A) = 1, n(A) = 0$, and $B(u)$ is a monomial with $N(B) \equiv n(B) = 0$, while $E(u)$ is a polynomial with $N(E) = m$ and $n(E) = 0$. We have a case where $N(A) - 1 \leq N(B)$ and $2N(C) - N(E) \leq N(B)$ from (11), so the general theory gives the following:

$$N(f) = N(h) = \frac{N(E)}{2} = \frac{m}{2}. \tag{21}$$

The reduced Equation (6) becomes the following:

$$-V(u+k)\frac{df}{du} + V\,f(u) + h(u) = 0. \tag{22}$$

An important remark is that $N(f)$ and $N(h)$ are multiples of $1/2$, integers only when m is even. The minimal degrees also satisfy the constraint (10) with $n(E) = 0$, so the following applies:

$$n(h) = -n(f). \tag{23}$$

As we consider $n(f) \geq 0$, we conclude that $n(h) \leq 0$. More precisely, $n(h) = 0$ if and only if $n(f) = 0$. The flow $f(u)$ to be attached to (19) will be given by (7), but its expansion essentially depends on the considered value of m. For convenience, we will choose the following notations:

$$f(u) = \alpha_{\frac{m}{2}} u^{\frac{m}{2}} + \alpha_{\frac{m-2}{2}} u^{\frac{m-2}{2}} + \ldots + \alpha_0;\ \text{for } m = even, \tag{24}$$
$$f(u) = \alpha_{\frac{m}{2}} u^{\frac{m}{2}} + \alpha_{\frac{m-2}{2}} u^{\frac{m-2}{2}} + \ldots + \alpha_1 u^{\frac{1}{2}};\ \text{for } m = odd. \tag{25}$$

Similarly, we will consider $h(u)$, given through the following:

$$h(u) = \beta_{\frac{m}{2}} u^{\frac{m}{2}} + \beta_{\frac{m-2}{2}} u^{\frac{m-2}{2}} + \ldots + \beta_0;\ \text{for } m = even, \tag{26}$$
$$h(u) = \beta_{\frac{m}{2}} u^{\frac{m}{2}} + \beta_{\frac{m-2}{2}} u^{\frac{m-2}{2}} + \ldots + \beta_{-1} u^{-\frac{1}{2}};\ \text{for } m = odd. \tag{27}$$

The previous relations have to make compatible Equations (5) and (6), equations that, for gDBM, have the following specific form:

$$p(u+k)^m + q = f(u)\,h(u), \tag{28}$$
$$-V(u+k)\frac{df}{du} + Vf(u) = h(u). \tag{29}$$

Using the suitable expressions of $f(u)$ and $h(u)$ from above in these last two equations and asking for the canceling of the coefficients of various powers of u, we get an algebraic system that allows us to determine the coefficients α_i and β_j. We will see that very important are the last coefficients, $\{\alpha_0, \beta_0\}$ for $m = even$ and, respectively, $\{\alpha_1, \beta_{-1}\}$ for $m = odd$, and the last equations lead to the following relations among them:

$$\beta_0 = -V\alpha_0; \beta_{-1} = Vk\frac{\alpha_1}{2}. \tag{30}$$

Conclusion: To make the large number of mathematical derivations clearer, we can synthesize the flowchart that is used in the attached flow method to solve the gDBM equation:

- We start from the gTDBM Equation (13) and transform it into (15) via an adequate change in the variable and choice of parameters.
- The PDE (15) is reduced to the ODE (16) using the wave variable.
- The translation (18) leads to Equation (19), which will be studied for various values of m.
- We apply the first specific ingredient of the attached flow approach: the decomposition of $E(u)$, as in (5). The gDBM Equation (19) takes the reduced form (22).

- We apply the second ingredient of the attached flow: a balancing procedure that leads to explicit forms of $f(u)$ and $h(u)$, which depend on m: (i) if m is even, with (24) and (26); (ii) if m is odd, with (25) and (27).
- We use these expressions to find the solutions of (19) for $m = 1, \ldots, 5$.

4.1. Solutions for $m = 1$

Let us consider first that $m = 1$, so the equation to be solved is as follows:

$$-V(u+k)u'' + Vu'^2 + pu + (pk+q) = 0. \tag{31}$$

In this case, using (25) and (27) in (28), respectively (29), will yield the following:

$$f(u) = \alpha_1 u^{\frac{1}{2}}, \quad h(u) = -\frac{V\alpha_1}{2}\left(u^{\frac{1}{2}} - ku^{-\frac{1}{2}}\right), \tag{32}$$

with the following compatibility condition:

$$k = -\frac{q}{2p}, \; p \neq 0. \tag{33}$$

In conclusion, the gDBM equation can be solved for $m = 1$ using the attached flow method. A translation with k, given by (33), is required. The equation accepts, in this case, a solution of the following form:

$$u(\xi) = -\frac{p}{4V}\xi^2 - \frac{q}{2p}. \tag{34}$$

We can see from (33) that $k = 0$ when $q = 0$. No translation is needed in this case. Otherwise, the values of k are influenced by the values of q and p. As $p \neq 0$ the specific form of the final parabolic solution (34) depends on the parametric errors induced via q and V.

4.2. Solutions for $m = 2$

When we consider $m = 2$ in (19), the equation we have to solve will be as follows:

$$-V(u+k)\,u'' + Vu'^2 + p(u+k)^2 + q = 0. \tag{35}$$

As $N(E) = 2 = even, n(E) = 0$, we have the following: $N(f) = N(h) = 1, n(f) = n(h) = 0$. A direct check shows that trying to apply the aforementioned solving procedure considering α_i as constants leads us to incompatibilities. It is not possible in this case to determine $f(u)$ and $h(u)$ in the form (7) with α_i and β_j as simple constants due to the fact that the first two terms, $-Vu\frac{df}{du}$ and Vf, from the reduced Equation (22) cancel each other and lead to $h = 0$, incompatible with the decomposition $fh = E \neq 0$. For the procedure to work, we must now consider a flow with an explicit dependence on the independent variable, $f = f(\xi, u(\xi))$, as in (8). We have to apply the method of the attached flow with variable coefficients for f and h. We will consider the following:

$$f = \alpha_1(\xi)u(\xi) + \alpha_0(\xi), \; h = \beta_1(\xi)u(\xi) + \beta_0(\xi). \tag{36}$$

Using these expressions in the decomposition $E = fh$ and in the reduced Equation (22), we generate a system of equations that allows, in principle, for finding the functions $\{\alpha_i(\xi), \beta_i(\xi), i = 0, 1\}$ and the compatible values of the parameters k, V, p, q. From the decomposition $E = fh$, we get relations allowing to express β_0, β_1 in terms of α_0, α_1:

$$\begin{aligned} fh &= \alpha_1\beta_1 u^2 + (\alpha_1\beta_0 + \alpha_0\beta_1)u + \alpha_0\beta_0, \\ fh &= E = p(u+k)^2 + q = pu^2 + 2pku + pk^2 + q. \end{aligned}$$

$$\begin{aligned}
\alpha_1\beta_1 &= p,\ \alpha_1\beta_0+\alpha_0\beta_1 = 2pk,\ \alpha_0\beta_0 = pk^2+q, \\
\alpha_1\beta_1 &= p \to \beta_1 = \frac{p}{\alpha_1}; \alpha_0\beta_0 = pk^2+q \to \beta_0 = \frac{pk^2+q}{\alpha_0}, \\
\alpha_1\beta_0+\alpha_0\beta_1 &= 2pk \to \frac{\alpha_1}{\alpha_0}(pk^2+q)+\frac{\alpha_0}{\alpha_1}p = 2pk.
\end{aligned} \tag{37}$$

Introducing these expressions in (35) and canceling the coefficients of various powers of u, we get the following compatibility conditions:

$$\begin{aligned}
\alpha_1 &= \frac{p}{V}\xi + C, \\
\alpha_0 &= \frac{kp}{V}\xi + 2kC = k\alpha_1, \\
q &= 0.
\end{aligned}$$

Conclusion: In the case of $m = 2$, the generalized DBM equation takes the form (35). It has a nontrivial solution for $q = 0$ and $k = 0$, with the corresponding attached flow that solves the equation being as follows:

$$f(\xi, u) = \frac{p}{V}\xi u(\xi) + \frac{kp}{V}\xi = \frac{p}{V}\xi u.$$

By integrating the flow equation, we get the following solution for the gDBM equation:

$$u(\xi) = Ce^{\frac{p\xi^2}{2V}}, C = const.$$

Remark: Numerical computations show that nontrivial solutions could be also obtained for $q \neq 0$, but around $\xi = 0$ only. Equation (35) could also admit a nontrivial solution for $q \neq 0$ if p and q also become functions of the independent variable ξ. In this case, the problem of the parametric errors becomes important and has to be properly considered.

4.3. Solutions for $m = 3$

In the case of $m = N(E) = 3 = odd$, using (10) and (11), we conclude that $N(f) = 3/2$ and $N(h) = 3/2$. For the minimal degrees, as $n(E) = 0$, we get $n(f) = 1/2$ and $n(h) = -1/2$. We are in a situation where the degrees of $f(u)$ and $h(u)$ are no longer integers. We will consider the following:

$$f(u) = \alpha_3 u^{\frac{3}{2}} + \alpha_1 u^{\frac{1}{2}},\ \ h(u) = \beta_3 u^{\frac{3}{2}} + \beta_1 u^{\frac{1}{2}} + \beta_{-1} u^{-\frac{1}{2}}.$$

We impose the compatibility conditions (28) and (29), and we get the following algebraic system:

$$\alpha_3\beta_3 = p,\ \alpha_3\beta_{-1} + \alpha_1\beta_1 = 3pk^2,$$

$$\alpha_3\beta_1 + \alpha_1\beta_3 = 3pk,\ \alpha_1\beta_{-1} = pk^3 + q$$

with the coefficients β given via the following:

$$\beta_3 = \frac{V\alpha_3}{2}; \beta_1 = \frac{3Vk}{2}\alpha_3 - \frac{V\alpha_1}{2}; \beta_{-1} = \frac{Vk\alpha_1}{2}.$$

These relations are compatible with the following:

$$k = \frac{1}{2}\left(\frac{q}{p}\right)^{\frac{1}{3}}. \tag{38}$$

We conclude that applying the attached flow method to the generalized DBM Equation (19) in the case of $m = 3$ asks for a translation, $u \to u + k$, with $k \neq 0$. Depending on the values of the other parameters, the following solutions can be generated:

(i) For $V > 0$ and $q \neq 0$, we get periodic solutions:

$$u_1(x,t) = \frac{1}{2}\left(\frac{q}{p}\right)^{\frac{1}{3}} + \frac{3}{2}\left(\frac{q}{p}\right)^{\frac{1}{3}} tg^2\left(\sqrt{\frac{3}{4V}}p^{\frac{1}{3}}q^{\frac{1}{6}}(x - Vt)\right),$$

$$u_2(x,t) = \left(\frac{q}{p}\right)^{\frac{1}{3}}\left[\frac{1}{2} + \frac{3}{2}\cot^2\left(-\sqrt{\frac{3}{4V}}p^{\frac{1}{3}}q^{\frac{1}{6}}(x - Vt)\right)\right];$$

(ii) For $V < 0$ and $q \neq 0$, the solutions are hyperbolic:

$$u_3(x,t) = \left(\frac{q}{p}\right)^{\frac{1}{3}}\left[\frac{1}{2} - \frac{3}{2}th^2\left(\sqrt{\frac{3}{4V}}p^{\frac{1}{3}}q^{\frac{1}{6}}(x - Vt)\right)\right],$$

$$u_4(x,t) = \left(\frac{q}{p}\right)^{\frac{1}{3}}\left[\frac{1}{2} + \frac{3}{2}cth^2\left(-\sqrt{\frac{3}{4V}}p^{\frac{1}{3}}q^{\frac{1}{6}}(x - Vt)\right)\right];$$

(iii) For $q = 0$, we get a rational solution:

$$u_5(x,t) = \frac{2V}{(x - Vt)^2}$$

Let us mention that all of these solutions were obtained and reported in [14] but using other numerical methods, not directly as here. The solutions are influenced by different choices of q and p. Other interesting numerical experiments on the DBM equation and the influence of the parameters on the solutions are included in [32,33]. As a numerical example, if $p < 0$ and $q = 1$, the hyperbolic solution could become a harmonic one, and vice versa. So, the parameter values can strongly affect the types of solutions.

4.4. Solutions for $m = 4$

Let now us consider Equation (19) with $m = 4$. The general relation (24) leads us to the idea that the flow should have the following form:

$$f(u) = \alpha_2 u^2 + \alpha_1 u + \alpha_0. \tag{39}$$

The compatibility of (28) and (29) can be obtained if and only if $k = 0$. The coefficients α_i from (39) satisfy, in this case, the following constraints:

$$V\alpha_2^2 = p, \alpha_1 = 0,\ V\alpha_0^2 = q.$$

As an explicit expression of the flow, we can use the following:

$$f(u) = \pm\sqrt{\frac{p}{V}}u^2 \pm \sqrt{-\frac{q}{V}}. \tag{40}$$

We can generate various numerical experiments using the previous expression to integrate the flow Equation (40):

- For $q \neq 0$ and $-\frac{pq}{V^2} > 0$, the solutions are periodic:

$$u_1(\xi) = \left(-\frac{q}{p}\right)^{\frac{1}{4}} + \left(-\frac{q}{p}\right)^{\frac{1}{4}} tg\left(-\frac{pq}{V^2}\right)^{\frac{1}{4}}\xi,$$

$$u_2(\xi) = \left(-\frac{q}{p}\right)^{\frac{1}{4}} + \left(-\frac{q}{p}\right)^{\frac{1}{4}}\cot\left[-\left(-\frac{pq}{V^2}\right)^{\frac{1}{4}}\xi\right]; \tag{41}$$

- For $q \neq 0$ and $-\frac{pq}{V^2} < 0$, the solutions are hyperbolic:

$$u_3(\xi) = \left(-\frac{q}{p}\right)^{\frac{1}{4}} + \left(-\frac{q}{p}\right)^{\frac{1}{4}} th\left(-\frac{pq}{V^2}\right)^{\frac{1}{4}}\xi,$$

$$u_4(\xi) = \left(-\frac{q}{p}\right)^{\frac{1}{4}} + \left(-\frac{q}{p}\right)^{\frac{1}{4}} cth\left[-\left(-\frac{pq}{V^2}\right)^{\frac{1}{4}}\xi\right]; \tag{42}$$

- For $q = 0$, the solution is rational:

$$u_5 = -\left(\frac{V}{p}\right)^{\frac{1}{2}} \frac{1}{\xi} \tag{43}$$

4.5. Solution for $m = 5$

Let us now consider that $m = 5$ in (19). We have, in the case of (25), the flow given by the following:

$$f(u) = \alpha_5 u^{\frac{5}{2}} + \alpha_3 u^{\frac{3}{2}} + \alpha_1 u^{\frac{1}{2}}.$$

The complementary function $h(u)$ will be as follows:

$$h(u) = \beta_5 u^{\frac{5}{2}} + \beta_3 u^{\frac{3}{2}} + \beta_1 u^{\frac{1}{2}} + \beta_{-1} u^{-\frac{1}{2}}.$$

The compatibility of the previous expressions with (28) leads to the following algebraic system:

$$\begin{gathered}
\alpha_5\beta_5 = p, \\
\alpha_5\beta_3 + \alpha_3\beta_5 = 5pk, \\
\alpha_5\beta_1 + \alpha_3\beta_3 + \alpha_1\beta_5 = 10pk^2, \\
\alpha_5\beta_{-1} + \alpha_3\beta_1 + \alpha_1\beta_3 = 10pk^3, \\
\alpha_3\beta_{-1} + \alpha_1\beta_1 + \alpha_2\beta_0 = 5pk^4, \\
\alpha_1\beta_{-1} = pk^5 + q.
\end{gathered} \tag{44}$$

The compatibility with (29) provides other supplementary requirements:

$$\begin{aligned}
\beta_5 &= \frac{3V\alpha_5}{2}, \\
\beta_3 &= \frac{5\alpha_5 Vk}{2} + \frac{V\alpha_3}{2}, \\
\beta_1 &= \frac{3\alpha_3 Vk}{2} - \frac{V\gamma\alpha_1}{2}, \\
\beta_{-1} &= \frac{\alpha_1}{2} Vk.
\end{aligned} \tag{45}$$

Direct computations show that the only case of the compatibility of (44) and (45) is for $q = 0$. In this case, $k = 0$, and the solution has the following form:

$$u(\xi) = \left(\frac{3p}{2V}\right)^{1/3} \xi^{2/3}.$$

Remark: Let us note that, as in the case of $m = 5$, for any $m > 5$ the gDBM equation may admit analytic solutions for $q = 0$ only. This is somehow related to the fact that the compatibility conditions that must be imposed lead to algebraic equations of orders higher than 5 and there are no general numerical formulas that provide analytic solutions expressed in radicals. In fact, as Abel and Ruffini established long ago [34,35], such formulas exist up to $m = 4$, not for quintic equations. These last equations could have solutions in terms of elliptic functions [36] or as very complicated expressions of no practical interest.

5. Comments on the Method and Results

This paper has dealt with applying the method of the attached flow proposed in [1] to a particular case of the reaction–diffusion Equation (1), namely the case of $C(u) = 0$. The key aspects of the attached flow method applied to a general equation of the form (1) is the decomposition $E(u) = f(u)h(u)$, where $f(u)$ represents the flow satisfying the equation $u' = f(u)$. The advantage of this decomposition is that it is always feasible, it brings the equation to the reduced form (6), and it allows the flow to be determined as polynomials of the form (7). The extreme values of the degrees, $n(f)$ and $N(f)$, are obtained via other important aspects of the method: the introduction of gradation rules and the requirement that terms of the same power in u compensate each other. The graduation rules seem to introduce restrictions, being applicable only to equations with polynomial coefficients, but most models of physical interest, defined in various space–time dimensions [37], belong to this category. More sophisticated graduations appear when multidifferential complexes are considered [38,39].

In the case of gDBM Equation (16), the balancing requirement leads to a system of algebraic equations that must be solved. The graduation rules show us that, for some values of m, compatibility problems could appear in $n(f)$, and as a novelty brought about via the paper, a translation, $u \to u + k$, was taken into account so that we studied Equation (19):

$$-V(u+k)u'' + Vu'^2 + p(u+k)^m + q = 0.$$

Moreover, only for $m = 2$, that is, the differentiability order of gDMB, the compatibility in the highest degree, $N(f)$, asks for a dependency of ξ in the flow equation, which could be written in the general form as follows:

$$u'(\xi) = f(\xi\delta_{m2}, u(\xi)),$$

where δ represents the Kronecker symbol.

The gDBM Equation (19) was written with five parameters: two real parameters, p and q, the wave velocity, V and two others of special interest in our study: m—the degree of nonlinearity of the free term, and the translation parameter k. Let us summarize the main solvable cases and the parametric error influence we have reported here, based on various numerical experiments. If we consider $q = 0$, the equation can be solved directly for any value of m, and its general solution does not depend on k. For $V \neq 0$, $p \neq 0$, the solution behaves as follows:

$$u(\xi) \sim \begin{cases} \xi^{-\frac{2}{m-2}}, \ m \neq 2 \\ e^{\frac{p}{2V}\xi^2}, m = 2 \end{cases}.$$

As we can see, a special situation appears for $m = 2$, and it is due to the fact that gDBM is a second-order differential equation. Trying to apply the attached flow in this case, we needed to consider flows of the form (8), depending explicitly on the independent variable, ξ.

For $q \neq 0$, depending on m, we obtained solutions with or without translations. Practically, no translation is necessary for $m = 2, 4$, and a translation, $k \neq 0$, is required for $m = 1, 3$. For $m = 5$, our method allowed us to find solutions in the case of $q = 0$ only. For $q \neq 0$, the system of algebraic equations to be solved contains an equation of the degree of 5, whose roots cannot be expressed with radicals. It cannot be compatibilized with the other equations of the system. This impossibility of compatibilization is valid for any $m > 6$, so all these equations cannot be analytically solved, not because of the attached flow method but, rather, due to the Abel–Ruffini theorem.

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