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A Comprehensive Study of the Langevin Boundary Value Problems with Variable Order Fractional Derivatives

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Abstract: The authors investigate Langevin boundary value problems containing a variable order Caputo fractional derivative. After presenting the background for the study, the authors provide the definitions, theorems, and lemmas that are required for comprehending the manuscript. The existence of solutions is proved using Schauder's fixed point theorem; the uniqueness of solutions is obtained by adding an additional hypothesis and applying Banach's contraction principle. An example is provided to demonstrate the results.

Keywords: variable order fractional operators; boundary value problem; Schauder's fixed point theorem; Banach's contraction principle; fractional Langevin equation

MSC: 34A08; 34A12; 26A33; 34B15

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1. Introduction

Fractional calculus is an intriguing facet of mathematical analysis that has garnered increasing attention across various scientific disciplines by extending differentiation and integration concepts to non-integer orders. While the usual calculus focuses on integer-order derivatives and integrals, fractional calculus expands these operations to include non-integer and even variable orders. This results in a more detailed and flexible description of physical processes that are especially useful in domains like physics, engineering, and biology (see [1–4]). Variable order fractional calculus is a key advancement in this discipline by offering the new concept that the order of differentiation or integration might be a variable rather than a fixed value. This breakthrough provides a strong tool for modeling complicated systems with various degrees of memory and non-local effects that results in a more realistic depiction of real-world processes and makes it a useful and adaptable tool (see [5–9] and the references therein).

Brownian motion is the random motion of particles suspended in a fluid (liquid or gas) resulting from their collision with fast molecules in the considered fluid. This phenomenon is named after the Scottish botanist Robert Brown, who first observed it in 1827 while studying pollen particles suspended in water [10]. However, the mathematical explanation and formalization of Brownian motion came later, through the work of Albert Einstein and the French mathematician Louis Bachelier. The particles' motion is characterized by random and erratic changes in direction and speed, therefore playing a crucial role in modeling various phenomena by simplifying the dynamics of real systems and providing a useful framework for understanding the behavior of particles in a fluctuating environment. It is widely used in various fields that involve random fluctuations such as physics, chemistry, biology, and finance.

In the context of physics, processes are often described mathematically using the Langevin equation, which is a stochastic differential equation that describes the motion of a particle undergoing Brownian motion under the influence of a random force. It is commonly used in the study of statistical mechanics and is named after the French physicist Paul Langevin [11], who, in 1908, developed the traditional version of this equation in terms of ordinary derivatives of the form

$$m\frac{d^2}{dt^2}\vartheta = \lambda\frac{d}{dt}\vartheta + \eta(t) = 0,$$

where m is the mass of the particle, λ is the friction coefficient, and $\eta(t)$ is a random force.

However, in complex media, this model did not seem to give an accurate representation of the dynamics of the system. In 1966, Kubo [12] proposed the extended Langevin equation in which a frictional memory kernel was included in the Langevin equation to represent the fractal and memory features. Mainardi et al. [13,14] developed the fractional Langevin equation in the 1990s. This yielded many interesting results regarding the existence, uniqueness, and stability of solutions of fractional order Langevin equations; for more details, see [15–17] and the references therein.

In [18], Abbas et al. discussed the solvability of the following Langevin equation with two Hadamard fractional derivatives

$$\begin{cases} {}^H\mathcal{D}^{\alpha}_{1,t}\left({}^H\mathcal{D}^{\beta}_{1,t} - \lambda\right)\vartheta(t) = H(t,\vartheta(t)), & t \in [1,e], \\ \left({}^H\mathcal{D}^{\beta}_{1,t} - \lambda\right)\vartheta(e) = 0, \quad {}^H I^{1-\beta}_{1^+}\vartheta(1) = c_0, \end{cases}$$

where $c_0 \in \mathbb{R}$, $\lambda > 0$, ${}^H\mathcal{D}^{\alpha}_{1,t}$, and ${}^H\mathcal{D}^{\beta}_{1,t}$ denote Hadamard fractional derivatives of orders α and β $(0 < \alpha,\ \beta \leq 1)$, respectively, ${}^H I^{1-\alpha}_{1^+}$ denotes the left hand Hadamard fractional integral of order $1-\alpha$, and $H : [1,e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. The approach they used involved the analysis of a Volterra integral equation and properties of the Mittag-Leffler function.

Recently, Hilal et al. [19] investigated the existence and uniqueness of solutions to the following boundary value problem for the Langevin equation with the Hilfer fractional derivative

$$\begin{cases} {}^H\mathcal{D}^{\alpha_1,\gamma_1}\left({}^H\mathcal{D}^{\alpha_2,\gamma_2} - \lambda\right)\vartheta(t) = H(t,\vartheta(t)), & a \leq t \leq b, \\ \vartheta(a) = 0, \quad \vartheta(b) = \sum_{i=1}^{n}\mu_i(I^{v_i}(\vartheta))(\eta), & a < \eta < b, \end{cases}$$

where ${}^H\mathcal{D}^{\alpha_i,\gamma_i}$, $i = 1,2$, are Hilfer fractional derivatives of order $0 < \alpha_i < 1$ and parameters $0 \leq \gamma_i \leq 1$, $\lambda \in \mathbb{R}$, $a \geq 0$, I^{v_i} is the Riemann–Liouville fractional integral of order $v_i > 0$, $\mu_i \in \mathbb{R}$, and $H : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

As far as we know, there are no contributions in the literature on the solutions of fractional Langevin equations of variable order.

In this paper, we investigate the Langevin boundary value problem involving variable order Caputo fractional derivatives

$$\begin{cases} {}^C\mathcal{D}^{\alpha(t)}_{0^+}\left({}^C\mathcal{D}^{\beta(t)}_{0^+} - \lambda\right)\vartheta(t) = H(t,\vartheta(t)), & t \in [0,T], \\ \vartheta(0) = \vartheta(T) = 0. \end{cases} \tag{1}$$

Here, $0 < \alpha(t),\ \beta(t) < 1$, $\lambda \in \mathbb{R}^+$, $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and ${}^C D^{\alpha(t)}_{0^+}$ and ${}^C D^{\beta(t)}_{0^+}$ are Caputo fractional derivatives of variable orders $\alpha(t)$ and $\beta(t)$, respectively, for the function ϑ. These are formally defined by (see [5])

$$\begin{aligned} {}^C\mathcal{D}^{\alpha(t)}_{0^+}\vartheta(t) &= \frac{1}{\Gamma(1-\alpha(t))}\int_0^t (t-s)^{-\alpha(t)}\vartheta'(s)ds, \quad t > 0, \\ {}^C\mathcal{D}^{\beta(t)}_{0^+}\vartheta(t) &= \frac{1}{\Gamma(1-\beta(t))}\int_0^t (t-s)^{-\beta(t)}\vartheta'(s)ds, \quad t > 0, \end{aligned}$$

The Riemann–Liouville integrals of ϑ of variable orders $\alpha(t)$ and $\beta(t)$, respectively, are given by (see, for example, [8])

$$I_{0^+}^{\alpha(t)}\vartheta(t) = \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1}\vartheta(s)ds, \quad t>0,$$
$$I_{0^+}^{\beta(t)}\vartheta(t) = \frac{1}{\Gamma(\beta(t))}\int_0^t (t-s)^{\beta(t)-1}\vartheta(s)ds, \quad t>0.$$

This paper is organized as follows. In Section 2, we present some definitions and necessary lemmas associated with variable order fractional Langevin boundary value problems. In Section 3, we establish the existence and uniqueness of solutions for the problem (1). In the last section, we present an example to illustrate the results we obtained.

2. Preliminaries

In this section, we introduce some fundamental concepts that will be needed for obtaining our results.

Let $[0,T]$, $T>0$, be a subset of $\mathbb{R}$. By $C([0,T],\mathbb{R})$, we mean the Banach space of continuous functions $\vartheta : [0,T] \to \mathbb{R}$ with the usual supremum norm

$$||\vartheta(t)||_\infty = \sup\{|\vartheta(t)|, \quad t \in [0,T]\},$$

and we let $L^1([0,T],\mathbb{R})$ be the Banach space of measurable functions $\vartheta : [0,T] \to \mathbb{R}$ that are Lebesgue integrable, equipped with the norm

$$||\vartheta||_{L^1} = \int_0^T |\vartheta(s)|\, ds.$$

Definition 1 ([20])**.** *Let S be a subset of $\mathbb{R}$.*

(i) *S is called a generalized interval if it is either a standard interval, a point, or $\emptyset$.*
(ii) *If S is a generalized interval, then the finite set $\mathcal{P}$ consisting of generalized intervals contained in S is called a partition of S provided that every $x \in S$ lies in exactly one of the generalized intervals in the finite set $\mathcal{P}$.*
(iii) *We say that the function $\psi : t \mapsto \mathbb{R}$ is piece-wise constant with respect to the partition $\mathcal{P}$ of S, i.e., for any $I \in \mathcal{P}$, ψ is constant on I.*

In what follows, $[\alpha]$ denotes the greatest integer function of α.

Lemma 1 ([1])**.** *Let $\alpha, \beta > 0$, $0 < a < b$, $\vartheta \in L^1(a,b)$, and ${}^C\mathcal{D}_{a^+}^{\alpha}\vartheta \in L^1(a,b)$. Then, the unique solution of the equation*

$${}^C\mathcal{D}_{a^+}^{\alpha}\vartheta(t) = 0,$$

is given by

$$\vartheta(t) = \varrho_0 + \varrho_1(t-a) + \varrho_2(t-a)^2 + \cdots + \varrho_{i-1}(t-a)^{i-1},$$

where $i = [\alpha]+1$ and $\varrho_k \in \mathbb{R}$, $k = 0,1,\ldots,i-1$. Moreover,

$$I_{a^+}^{\varpi_1}\, {}^C\mathcal{D}_{a^+}^{\alpha}\vartheta(t) = \vartheta(t) + \varrho_0 + \varrho_1(t-a) + \varrho_2(t-a)^2 + \cdots + \varrho_{i-1}(t-a)^{i-1},$$

$${}^C\mathcal{D}_{a^+}^{\alpha}\, I_{a^+}^{\alpha}\vartheta(t) = \vartheta(t),$$

and

$$I_{a^+}^{\alpha} I_{a^+}^{\beta}\vartheta(t) = I_{a^+}^{\beta} I_{a^+}^{\alpha}\vartheta(t) = I_{a^+}^{\alpha+\beta}\vartheta(t).$$

The following result is known as Schauder's fixed point theorem.

Theorem 1 (([1] [Theorem 1.7]) [21])**.** *Let E be a Banach space, B be a nonempty bounded convex and closed subset of E, and $\mathcal{G} : B \to B$ be a compact and continuous map. Then, $\mathcal{G}$ has at least one fixed point in B.*

3. Existence of Solutions

Based on the previous discussion, in this section, we present our main results.

Let $\mathcal{P} = \{[0, T_1], (T_1, T_2], (T_2, T_3], \ldots, (T_{n-1}, T]\}$ be a partition of the finite interval $[0, T]$, and let $\alpha : [0, T] \to (0, 1]$, and $\beta : [0, T] \to (0, 1]$ be two piecewise constant functions with respect to $\mathcal{P}$ given by

$$\alpha(t) = \sum_{i=1}^{n} \alpha_i \mathbb{I}_i(t) = \begin{cases} \alpha_1, & t \in [0, T_1], \\ \alpha_2, & t \in (T_1, T_2], \\ \vdots \\ \alpha_n, & t \in (T_{n-1}, T], \end{cases} \qquad \beta(t) = \sum_{i=1}^{n} \beta_i \mathbb{I}_i(t) = \begin{cases} \beta_1, & t \in [0, T_1], \\ \beta_2, & t \in (T_1, T_2], \\ \vdots \\ \beta_n, & t \in (T_{n-1}, T], \end{cases}$$

where $0 < \alpha_i, \beta_i < 1$, $i \in \{1, 2, \ldots, n\}$, are constants, and $\mathbb{I}_i$ is the characteristic function for the interval $(T_{i-1}, T_i]$ for each $i \in \{1, 2, \ldots, n\}$, i.e.,

$$\mathbb{I}_i(t) = \begin{cases} 1, & t \in \mathcal{J}_i, \\ 0, & \text{elsewhere.} \end{cases}$$

Hence, we obtain

$$\begin{aligned} &{}^{C}\mathcal{D}_{0^+}^{\alpha(t)} \left({}^{C}\mathcal{D}_{0^+}^{\beta(t)} - \lambda \right) \vartheta(t) \\ &\quad = \int_0^t \frac{(t-s)^{-\sum_{i=1}^{n} \alpha_i \mathbb{I}_i(t)}}{\Gamma(1 - \sum_{i=1}^{n} \alpha_i \mathbb{I}_i(t))} \frac{d}{ds} \left(\int_0^s \frac{(s-w)^{-\sum_{i=1}^{n} \beta_i \mathbb{I}_i(s)}}{\Gamma(1 - \sum_{i=1}^{n} \beta_i \mathbb{I}_i(s))} \vartheta'(w) dw - \lambda \vartheta(s) \right) ds. \end{aligned}$$

The equation in the problem (1) can then be written as

$$\int_0^t \frac{(t-s)^{-\sum_{i=1}^{n} \alpha_i \mathbb{I}_i(t)}}{\Gamma(1 - \sum_{i=1}^{n} \alpha_i \mathbb{I}_i(t))} \frac{d}{ds} \left(\int_0^s \frac{(s-w)^{-\sum_{i=1}^{n} \beta_i \mathbb{I}_i(s)}}{\Gamma(1 - \sum_{i=1}^{n} \beta_i \mathbb{I}_i(s))} \vartheta'(w) dw - \lambda \vartheta(s) \right) ds = H(t, \vartheta(t))$$

for $0 \le t \le T < +\infty$. We denote by $E_i = C([T_{i-1}, T_i], \mathbb{R})$, the class of functions that form a Banach space with the norm

$$||\vartheta||_{E_i} = \sup_{t \in [T_{i-1}, T_i]} |\vartheta(t)|, \quad i \in \{1, 2, \ldots, n\}.$$

Let the functions $\widehat{\vartheta}_i \in E_i$ be such that $\widehat{\vartheta}_i(t) = 0$ for all $t \in [0, T_{i-1}]$ and all $i \in \{2, \ldots, n\}$. Therefore, in the interval $[0, T_1]$, we have

$${}^{C}\mathcal{D}_{0^+}^{\alpha_1} \left({}^{C}\mathcal{D}_{0^+}^{\beta_1} - \lambda \right) \widehat{\vartheta}(t) = \int_0^t \frac{(t-s)^{-\alpha_1}}{\Gamma(1-\alpha_1)} \frac{d}{ds} \left(\int_0^s \frac{(s-w)^{-\beta_1}}{\Gamma(1-\beta_1)} \widehat{\vartheta}'(w) dw - \lambda \widehat{\vartheta}(s) \right) ds = H(t, \widehat{\vartheta}(t)). \quad (2)$$

Again, in the interval $(T_1, T_2]$,

$${}^{C}\mathcal{D}_{T_1^+}^{\alpha_2} \left({}^{C}\mathcal{D}_{T_1^+}^{\beta_2} - \lambda \right) \widehat{\vartheta}(t) = \int_{T_1}^t \frac{(t-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} \frac{d}{ds} \left(\int_{T_1}^s \frac{(s-w)^{-\beta_2}}{\Gamma(1-\beta_2)} \widehat{\vartheta}'(w) dw - \lambda \widehat{\vartheta}(s) \right) ds = H(t, \widehat{\vartheta}(t)). \quad (3)$$

Similarly, in $(T_{i-1}, T_i]$,

$$\begin{aligned} {}^{C}\mathcal{D}_{T_{i-1}^+}^{\alpha_i} \left({}^{C}\mathcal{D}_{T_{i-1}^+}^{\beta_i} - \lambda \right) \widehat{\vartheta}(t) &= \int_{T_{i-1}}^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} \frac{d}{ds} \left(\int_{T_{i-1}}^s \frac{(s-w)^{-\beta_i}}{\Gamma(1-\beta_i)} \widehat{\vartheta}'(w) dw - \lambda \widehat{\vartheta}(s) \right) ds \\ &= H(t, \widehat{\vartheta}(t)). \end{aligned} \quad (4)$$

Thus, for each $i \in \{1, 2, \ldots, n\}$, we consider the auxiliary constant order boundary value problem

$$\begin{cases} {}^C\mathcal{D}^{\alpha_i}_{T^+_{i-1}} \left({}^C\mathcal{D}^{\beta_i}_{T^+_{i-1}} - \lambda \right) \widehat{\vartheta}(t) = H(t, \widehat{\vartheta}(t)), \qquad T_{i-1} < t \leq T_i, \\ \widehat{\vartheta}(T_{i-1}) = \widehat{\vartheta}(T_i) = 0. \end{cases} \tag{5}$$

Next, we define what we mean by a solution of (1).

Definition 2. *We say that the problem (1) has a solution $\vartheta \in C([0, T], \mathbb{R})$, if there exist functions ϑ_i, such that: $\vartheta_1 \in E_1$ satisfies Equation (2) with $\vartheta_1(0) = \vartheta_1(T_1) = 0$; $\vartheta_2 \in E_2$ satisfies Equation (3) with $\vartheta_2(T_1) = \vartheta_2(T_2) = 0$; $\vartheta_i \in E_i$ satisfies Equation (4) with $\vartheta_i(T_{i-1}) = \vartheta_i(T_i) = 0$ for $i \in \{3, \ldots, n\}$.*

Remark 1. *We say that problem (1) has a unique solution in $C([0, T], \mathbb{R})$ if the functions $\widehat{\vartheta}_i$ are unique for each $i \in \{1, 2, \ldots, n\}$.*

Based on the previous discussion, we have the following results.

Lemma 2. *Let $i \in \{1, \ldots, n\}$. Then, the function $\widehat{\vartheta}$ is a solution of (5) if and only if $\widehat{\vartheta}$ is a solution of the integral equation*

$$\begin{aligned} \widehat{\vartheta}(t) = &- \left(\lambda I^{\beta_i}_{T^+_{i-1}} \widehat{\vartheta}(t) - I^{\alpha_i + \beta_i}_{T^+_{i-1}} H(t, \widehat{\vartheta}(t)) \right) \Big|_{t = T_i} \left(\frac{t - T_{i-1}}{T_i - T_{i-1}} \right)^{\beta_i} + \frac{\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^t (t-s)^{\beta_i - 1} \widehat{\vartheta}(s) ds \\ &+ \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^t (t-s)^{\alpha_i + \beta_i - 1} H(s, \widehat{\vartheta}(s)) ds, \end{aligned} \tag{6}$$

for $t \in (T_{i-1}, T_i]$ for each $i \in \{1, 2, \ldots, n\}$.

Proof. Assume that $\widehat{\vartheta}$ satisfies (5). We transform (5) into an equivalent integral equation as follows. Let $t_{i-1} < t \leq t_i$; then, Lemma 1 implies

$$\left({}^C\mathcal{D}^{\beta_i}_{T^+_{i-1}} - \lambda \right) \widehat{\vartheta}(t) = I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{\vartheta}(t)) + \varrho_1,$$

so

$$\widehat{\vartheta}(t) = \lambda I^{\beta_i}_{T^+_{i-1}} \vartheta(t) + I^{\alpha_i + \beta_i}_{T^+_{i-1}} H(t, \widehat{\vartheta}(t)) + \frac{\varrho_1}{\Gamma(\beta_i + 1)} (t - T_{i-1})^{\beta_i} + \varrho_2.$$

Using the boundary conditions $\widehat{\vartheta}(T_i) = \widehat{\vartheta}(T_{i-1}) = 0$, we obtain

$$\begin{cases} \varrho_2 = 0 \\ \varrho_1 = -\dfrac{\Gamma(\beta_i + 1)}{(T_i - T_{i-1})^{\beta_i}} \left(\lambda I^{\beta_i}_{T^+_{i-1}} \widehat{\vartheta}(t) + I^{\alpha_i + \beta_i}_{T^+_{i-1}} H(t, \widehat{\vartheta}(t)) \right) \Big|_{t = T_i}. \end{cases}$$

Therefore, the solution of the auxiliary boundary value problem (5) is given by

$$\begin{aligned} \widehat{\vartheta}(t) = &- \left(\lambda I^{\beta_i}_{T^+_{i-1}} \widehat{\vartheta}(t) + I^{\alpha_i + \beta_i}_{T^+_{i-1}} H(t, \widehat{\vartheta}(t)) \right) \Big|_{t = T_i} \left(\frac{t - T_{i-1}}{T_i - T_{i-1}} \right)^{\beta_i} + \frac{\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^t (t-s)^{\beta_i - 1} \widehat{\vartheta}(s) ds \\ &+ \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^t (t-s)^{\alpha_i + \beta_i - 1} H(s, \widehat{\vartheta}(s)) ds. \end{aligned} \tag{7}$$

A straightforward calculation shows that if $\widehat{\vartheta}$ is given by (6), then it is a solution of (5) for each $i \in \{1, 2, \ldots, n\}$. $\square$

Before presenting our main results, we first state the following hypotheses that will be needed:

Hypothesis 1. *Let $H : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that there exist positive constants L_1 and L_2 such that*

$$|H(t, x)| \leq L_1 |x| + L_2$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.

Hypothesis 2. *The parameter λ satisfies*

$$\frac{\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{L_1}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} < \frac{1}{2}.$$

Theorem 2. *Assume that (H1) and (H2) hold. Then, the boundary value problem* (1) *has at least one solution in $C([0,T],\mathbb{R})$.*

Proof. Consider the mapping $\mathcal{G} : E_i \to E_i$ given by

$$\begin{aligned}(\mathcal{G}\widehat{\vartheta})(t) = &-\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{\vartheta}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}} H(t,\widehat{\vartheta}(t))\right)\Big|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i - T_{i-1}}\right)^{\beta_i} + \frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}\widehat{\vartheta}(s)ds \\ &+ \frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,\widehat{\vartheta}(s))ds.\end{aligned}$$

Let the ball $B_{R_i} = \{\vartheta \in E_i : ||\vartheta||_{E_i} \le R_i\}$ be a non-empty, closed, bounded, convex subset of E_i, where

$$R_i \ge \frac{\frac{2L_2}{\Gamma(\alpha_i+\beta_i+1)}(t_i - t_{i-1})^{\alpha_i+\beta_i}}{1-\left(\frac{2\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{2L_1}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i}\right)}.$$

The proof will be given through several steps.

Step 1: *For each $i \in \{1,2,\ldots,n\}$, $\mathcal{G}(B_{R_i}) \subset B_{R_i}$.* We have

$$\begin{aligned}|(\mathcal{G}\widehat{\vartheta})(t)| =&\Bigg|-\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{\vartheta}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}} H(t,\widehat{\vartheta}(t))\right)\Big|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i - T_{i-1}}\right)^{\beta_i} + \frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}\widehat{\vartheta}(s)ds \\ &+\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,\widehat{\vartheta}(s))ds\Bigg| \\ &\le \frac{2\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\beta_i-1}|\widehat{\vartheta}(s)|ds + \frac{2}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}|H(s,\widehat{\vartheta}(s))|ds \\ &\le \frac{2\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\beta_i-1}|\widehat{\vartheta}(s)|ds + \frac{2}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}(L_1|\widehat{\vartheta}(s)| + L_2)ds \\ &\le \frac{2\lambda||\widehat{\vartheta}||_{E_i}}{\Gamma(\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\beta_i-1}ds + \frac{2L_1||\widehat{\vartheta}||_{E_i}}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}ds \\ &+\frac{2L_2}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}ds \\ &\le \frac{2\lambda||\widehat{\vartheta}||_{E_i}}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{2L_1||\widehat{\vartheta}||_{E_i}}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} \\ &+\frac{2L_2}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} \\ &\le R_i.\end{aligned}$$

Step 2: *$\mathcal{G}$ is continuous for each $i \in \{1,2,\ldots,n\}$.* Let $\{\widehat{\vartheta}_m\}$ be a sequence such that $\widehat{\vartheta}_m \to \widehat{\vartheta}$ in B_{R_i}. Then, for each $t \in [T_{i-1}, T_i]$, $i \in \{1,2,\ldots,n\}$,

$$
\begin{aligned}
|(\mathcal{G}\widehat{\vartheta}_m)(t)-(\mathcal{G}\widehat{\vartheta})(t)| = & \Bigg| -\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{\vartheta}_m(t)+I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{\vartheta}_m(t))\right)\Bigg|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\\
&+\frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}\widehat{\vartheta}_m(s)ds\\
&+\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,\widehat{\vartheta}_m(s))ds\\
&+\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{\vartheta}(t)-I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{\vartheta}(t))\right)\Bigg|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\\
&-\frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}\widehat{\vartheta}(s)ds\\
&-\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,\widehat{\vartheta}(s))ds\Bigg|\\
\leq & \frac{2\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\beta_i-1}|\widehat{\vartheta}_m(s)-\widehat{\vartheta}(s)|ds\\
&+\frac{2}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}|H(s,\widehat{\vartheta}_m(s))-H(s,\widehat{\vartheta}(s))|ds.
\end{aligned}
$$

Taking into account the convergence of the sequence $\{\widehat{\vartheta}_m\}$, and the continuity of the function H, the right-hand side of the above inequality tends to zero as $m\to+\infty$. Therefore,

$$||\mathcal{G}(\widehat{\vartheta}_m)(t)-\mathcal{G}(\widehat{\vartheta})(t)||_{E_i}\to 0 \quad \text{as} \quad m\to+\infty.$$

Step 3: *$\mathcal{G}$ is relatively compact for each $i\in\{1,2,\dots,n\}$.* In view of **Step 1**, we have that $\mathcal{G}(B_{R_i})\subset B_{R_i}$. Thus, $\mathcal{G}(B_{R_i})$ is uniformly bounded. It remains to show that $\mathcal{G}$ is equicontinuous for each $i\in\{1,2,\dots,n\}$.

Let t_1, $t_2\in(T_{i-1},T_i]$. Then,

$$
\begin{aligned}
&|(\mathcal{G}\widehat{\vartheta})(t_1)-(\mathcal{G}\widehat{\vartheta})(t_2)|\\
&\quad= \Bigg| -\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{\vartheta}(t)+I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{\vartheta}(t))\right)\Bigg|_{t=T_i}\left(\frac{t_1-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\\
&\quad+\frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t_1}(t_1-s)^{\beta_i-1}\widehat{\vartheta}(s)ds\\
&\quad+\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t_1}(t_1-s)^{\alpha_i+\beta_i-1}H(s,\widehat{\vartheta}(s))ds\\
&\quad+\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{\vartheta}(t)+I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{\vartheta}(t))\right)\Bigg|_{t=T_i}\left(\frac{t_2-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\\
&\quad-\frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t_2}(t_2-s)^{\beta_i-1}\widehat{\vartheta}(s)ds\\
&\quad-\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t_2}(t_2-s)^{\alpha_i+\beta_i-1}H(s,\widehat{\vartheta}(s))ds\Bigg|\\
&\quad\leq \left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{\vartheta}(t)-I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{\vartheta}(t))\right)\Bigg|_{t=T_i}\left[\left(\frac{t_1-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}-\left(\frac{t_2-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\right]\\
&\quad+\frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t_1}\left[(t_1-s)^{\beta_i-1}-(t_2-s)^{\beta_i-1}\right]|\widehat{\vartheta}(s)|ds-\frac{\lambda}{\Gamma(\beta_i)}\int_{t_1}^{t_2}(t_2-s)^{\beta_i-1}|\widehat{\vartheta}(s)|ds\\
&\quad\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t_1}\left[(t_1-s)^{\alpha_i+\beta_i-1}-(t_2-s)^{\alpha_i+\beta_i-1}\right]|H(s,\widehat{\vartheta}(s))|ds\\
&\quad-\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{t_1}^{t_2}(t_2-s)^{\alpha_i+\beta_i-1}|H(s,\widehat{\vartheta}(s))|ds
\end{aligned}
$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. Hence, the mapping $\mathcal{G}$ is equicontinuous. Therefore, in view of the Ascoli–Arzelà Theorem, the mapping $\mathcal{G}$ is relatively compact on B_{R_i}.

It follows from Theorem 1 that the auxiliary boundary value problem (5) has at least one solution in B_{R_i} for each $i \in \{1, 2, \dots, n\}$.

As a result, the boundary value problem (1) has a least one solution in $C([0, T], \mathbb{R})$, which is given by

$$\vartheta(t) = \begin{cases} \vartheta_1(t) = \quad \widehat{\vartheta}_1(t), & t \in [0, T_1], \\ \vartheta_2(t) = \begin{cases} 0, & t \in [0, T_1], \\ \widehat{\vartheta}_2(t), & t \in (T_1, T_2], \end{cases} \\ \quad \vdots \\ \vartheta_n(t) = \begin{cases} 0, & t \in [0, T_{n-1}], \\ \widehat{\vartheta}_n(t), & t \in (T_{n-1}, T]. \end{cases} \end{cases}$$

□

In order to prove the uniqueness of solutions, we need to introduce an additional hypothesis:

Hypothesis 3. *There exists a positive constant L_3 such that $|H(t, x) - H(t, y)| \leq L_3|x - y|$ for each $t \in [0, T]$ and $x, y \in \mathbb{R}$.*

Theorem 3. *Assume that conditions (H1)–(H3) hold. Then, (5) has a unique solution on $[T_{i-1}, T_i]$ for each $i \in \{1, 2, \dots, n\}$, provided that*

$$\left(\frac{2\lambda}{\Gamma(\beta_i + 1)} (T_i - T_{i-1})^{\beta_i} + \frac{2L_3}{\Gamma(\alpha_i + \beta_i + 1)} (T_i - T_{i-1})^{\alpha_i + \beta_i} \right) < 1. \tag{8}$$

Proof. As previously shown in **Step 1** of the proof of Theorem 2, the mapping $\mathcal{G} : B_{R_i} \to B_{R_i}$ is uniformly bounded. It remains to show that $\mathcal{G}$ is a contraction.

Let $i \in \{1, 2, \dots, n\}$ and let ϑ_i, $\vartheta_i^* \in B_{R_i}$. Then,

$$\begin{aligned} |(\mathcal{G}\vartheta_i)(t) - (\mathcal{G}\vartheta_i^*)(t)| = \Bigg| &- \left(\lambda I_{T_{i-1}^+}^{\beta_i} \vartheta_i(t) + I_{T_{i-1}^+}^{\alpha_i+\beta_i} H(t, \vartheta_i(t)) \right)\Bigg|_{t=T_i} \left(\frac{t - T_{i-1}}{T_i - T_{i-1}} \right)^{\beta_i} \\ &+ \frac{\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^t (t - s)^{\beta_i - 1} \vartheta_i(s) ds \\ &+ \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^t (t - s)^{\alpha_i + \beta_i - 1} H(s, \vartheta_i(s)) ds \\ &+ \left(\lambda I_{T_{i-1}^+}^{\beta_i} \vartheta_i^*(t) + I_{T_{i-1}^+}^{\alpha_i+\beta_i} H(t, \vartheta_i^*(t)) \right)\Bigg|_{t=T_i} \left(\frac{t - T_{i-1}}{T_i - T_{i-1}} \right)^{\beta_i} \\ &- \frac{\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^t (t - s)^{\beta_i - 1} \vartheta_i^*(s) ds \\ &- \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^t (t - s)^{\alpha_i + \beta_i - 1} H(s, \vartheta_i^*(s)) ds \Bigg| \\ \leq\ & \frac{2\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{\beta_i - 1} |\vartheta_i(s) - \vartheta_i^*(s)| ds \\ &+ \frac{2}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{\alpha_i + \beta_i - 1} |H(s, \vartheta_i(s)) - H(s, \vartheta_i^*(s))| ds \\ \leq\ & \frac{2\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{\beta_i - 1} |\vartheta_i(s) - \vartheta_i^*(s)| ds \\ &+ \frac{2L_3}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{\alpha_i + \beta_i - 1} |\vartheta_i(s) - \vartheta_i^*(s)| ds \\ \leq\ & \left(\frac{2\lambda}{\Gamma(\beta_i + 1)} (T_i - T_{i-1})^{\beta_i} + \frac{2L_3}{\Gamma(\alpha_i + \beta_i + 1)} (T_i - T_{i-1})^{\alpha_i + \beta_i} \right) ||\vartheta_i - \vartheta_i^*||_{E_i}. \end{aligned}$$

In view of (8), $\mathcal{G}$ is a contraction for each $i \in \{1, 2, \dots, n\}$. As a consequence of Banach's fixed point theorem, the operator $\mathcal{G}$ has a unique fixed point, which corresponds to a unique solution of (5) on $(T_{i-1}, T_i]$ for each $i \in \{1, 2, \dots, n\}$. In view of Remark 1, we have the uniqueness of solutions to (1). □

4. Example

In this section, we illustrate the applicability of the results obtained in this paper. Consider the fractional Langevin boundary value problem

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha(t)}\left({}^{C}\mathcal{D}_{0^+}^{\beta(t)} - \frac{1}{8}\right)\vartheta(t) = \sin\left(\frac{1}{8}\vartheta(t)\right) + \frac{1}{(t+2)^2}, \quad 0 \le t \le 4, \\ \vartheta(0) = 0,\ \vartheta(4) = 0. \end{cases} \tag{9}$$

Here, $\lambda = \frac{1}{8}$, $H(t,x) = \sin\left(\frac{x}{8}\right) + \frac{1}{(t+2)^2}$, $T_1 = 2$, and $T_2 = 4$ so that our partition of $[0,4]$ becomes $\mathcal{P} = \{[0,2], (2,4]\}$. We take

$$\alpha(t) = \begin{cases} \frac{2}{10}, & t \in [0,2], \\ \\ \frac{7}{10}, & t \in (2,4], \end{cases} \qquad \beta(t) = \begin{cases} \frac{3}{10}, & t \in [0,2], \\ \\ \frac{5}{10}, & t \in (2,4]. \end{cases}$$

Since $|H(t,x)| \le \frac{1}{8}|x| + \frac{1}{4}$, in view of (H1), we see that $L_1 = \frac{1}{8}$ and $L_2 = \frac{1}{4}$. Consider the auxiliary boundary value problems

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\frac{2}{10}}\left({}^{C}\mathcal{D}_{0^+}^{\frac{3}{10}} + \frac{1}{8}\right)\vartheta(t) = \sin\left(\frac{1}{8}\vartheta(t)\right) + \frac{1}{(t+2)^2}, \quad 0 \le t \le 2, \\ \vartheta(1) = 0,\ \vartheta(2) = 0, \end{cases}$$

and

$$\begin{cases} {}^{C}\mathcal{D}_{2^+}^{\frac{7}{10}}\left({}^{C}\mathcal{D}_{2^+}^{\frac{5}{10}} + \frac{1}{8}\right)\vartheta(t) = \sin\left(\frac{1}{8}\vartheta(t)\right) + \frac{1}{(t+2)^2}, \quad 2 < t \le 4, \\ \vartheta(2) = 0,\ \vartheta(4) = 0. \end{cases}$$

Now, for $i \in \{1,2\}$, we have

$$\begin{cases} \frac{\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{L_1}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} \approx 0.37093 < \frac{1}{2}, \\ \\ \frac{\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{L_1}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} \approx 0.47880 < \frac{1}{2}. \end{cases}$$

so (H2) is satisfied. Therefore, by Theorem 2, the problem (9) has at least one solution given by

$$\vartheta(t) = \begin{cases} \vartheta_1(t) = \widehat{\vartheta}_1(t), & t \in [0,2], \\ \vartheta_2(t) = \begin{cases} 0, & t \in [0,2], \\ \widehat{\vartheta}_2(t), & t \in (2,4]. \end{cases} \end{cases}$$

To illustrate Theorem 3, in the above problem take $H(t,x) = \frac{|x|}{(10+t^2)(1+|x|)}$ for all $t \in [0,4]$. Clearly, $|H(t,x) - H(t,y)| \le \frac{1}{10}|x-y|$. Thus, (H3) is satisfied with $L_3 = \frac{1}{10}$.

Direct computations give

$$\begin{cases} \frac{2\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{2L_3}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} \approx 0.50250 < 1, \\ \\ \frac{2\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{2L_3}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} \approx 0.60743 < 1. \end{cases}$$

for $i \in \{1,2\}$, which proves the theorem.

5. Conclusions

In this paper we studied Langevin boundary value problems that contain a variable order Caputo fractional derivative. The background and motivation for the study were presented along with the concepts needed in the work. The existence of solutions was proved by applying Schauder's fixed point theorem, and the uniqueness of solutions was obtained by adding an additional hypothesis and applying Banach's contraction principle. To illustrate the applicability of the results, an example was given.

One possible direction for future research would be to try to apply different fixed-point theorems so that different assumptions to guarantee existence could be obtained. For example, it would be interesting to see if the Leggett and Williams fixed point theorem might be used. Another possible direction for future research could be to change the setting to Hölder-type spaces. An appropriate question to raise is whether it would be beneficial to use regulated functions rather than piecewise continuous functions.

The question of what is (are) the appropriate space (spaces) to use as a setting for fractional problems is an interesting one that is deserving of future investigation. The answer often depends on the type of fractional derivative involved and the form of the boundary conditions. The reader who is interested in this may wish to consult the monographs by Kilbas et al. [1] or Podlubny [4] for additional observations on this question.

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