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Hyperholomorphicity by Proposing the Corresponding Cauchy–Riemann Equation in the Extended Quaternion Field

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Abstract: In algebra, the sedenions, an extension of the octonion system, form a 16-dimensional noncommutative and nonassociative algebra over the real numbers. It can be expressed as two octonions, and a function and differential operator can be defined to treat the sedenion, expressed as two octonions, as a variable. By configuring elements using the structure of complex numbers, the characteristics of octonions, the stage before expansion, can be utilized. The basis of a sedenion can be simplified and used for calculations. We propose a corresponding Cauchy–Riemann equation by defining a regular function for two octonions with a complex structure. Based on this, the integration theorem of regular functions with a sedenion of the complex structure is given. The relationship between regular functions and holomorphy is presented, presenting the basis of function theory for a sedenion of the complex structure.

Keywords: sedenion; octonion; Cauchy–Riemann system; regular function

MSC: 32A99; 30G35; 32W50

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1. Introduction

Applying the Cayley–Dickson construction to octonions yields sedenions. In algebra, sedenions are hypercomplex numbers that belong to a 16-dimensional algebra over the real numbers. However, unlike octonions, sedenions are not an alternative algebra. In 1898, Conway and Smith [1,2] showed that algebra has a multiplicative for the reals, the complex numbers, the quaternions, and the octonions, which are processed by the usual Cayley–Dickson construction. They researched various operators on 16-dimensional real space to construct the multiplication and the Euclidean norm. Carmody [3] studied the arithmetic of sedenions with 16-dimensional hypercomplex numbers, and subsequently K. Imaeda and M. Imaeda [4] and Kvplinger [5] developed concepts of sedenion represented in rectangular form and defined a single norm and some notations into polar forms. They are classified in sedenions obtained by the Cayley–Dickson construction and conic sedenions belonging to a part of a hypernumber concept. Kim et al. [6,7] defined hyperholomorphic functions with dual sedenion variables and researched the properties of hyperholomorphic functions of dual sedenion variables in Clifford analysis by the condition of harmonicity.

By defining complex numbers as ordered pairs of real numbers, various mathematical results were produced. Inspired by this, quaternions were defined as ordered pairs of complex numbers. By presenting the operations and function theories of quaternions in the form of complex numbers, the properties and characteristics of complex numbers can be used to deal with the function theory of quaternions (see [8,9]). In addition, by defining an octonion as an ordered pair of quaternions, the functional theory of octonions was expanded by applying the structural characteristics of complex numbers and the properties of quaternions (see [10–13]). Concerning this approach, we define sedenions as ordered pairs of octonion numbers. Through this, we would like to see whether it is possible to arrive at a theory similar to the mathematical results of quaternions and octonions that is aimed at the structure of complex numbers. Our objective is to highlight the advantages of complex number structures and operations, and to simplify the base numbers so that operations can be handled more effectively. We can use the composition and characteristics of previous number systems in our analysis. However, operation laws such as commutative laws and combination laws cannot be organized with existing general operators. Therefore, we need to define operators that can be used in a number system that deals with the combination of complex structures. Two types of differential operators can be defined using multiplication defined on the sedenion number of a complex structure using two octonion numbers. It can express one Laplace operator and define a regular function from each differential operator. To perform an analytical approach to the complex sedenions number, a function is defined that has the complex sedenions number as a variable. We define differentiation, which is the basic operation for functions, and propose a tool to determine whether the defined differentiation is differentiable. In the case of complex numbers, the regular function is determined using the Cauchy–Riemann equation. For quaternions and octonions expressed in the form of complex numbers, the construction and alternative of the Cauchy–Riemann equation, which has an equivalence relationship with the normal function defined in each number system, was presented. We also propose a corresponding Cauchy–Riemann equation for defining a function with a expressed in the structure of a complex number as a variable. This defined function can be called a regular function. Using the definition of a regular function and the corresponding Cauchy–Riemann equation, the integration theorem in the complex sedenion system is presented. Additionally, the relationship between regularity and regular functions is proposed. It has been confirmed that the relationship between the integral theorem and regularity can be dealt with using a regular function of the hexadecimal number in a complex form. This also suggests that the contents of other function theories can be addressed in the future.

Section 2 presents the elements and functions of sedenions, denoted by $\mathbb{S}$, with two octonions, written as $\mathbb{O} \times \mathbb{O}$. Section 3 gives an extension to sedenions of the Cauchy–Riemann equations, which is similar to the construction of quaternions and octonions. Also, we define the regularity of a function and investigate some properties of regular functions with values in $\mathbb{O} \times \mathbb{O}$.

2. Preliminaries

The Sedenions, denoted by $\mathbb{S}$, are a non-commutative, 16-dimensional $\mathbb{R}$-field consisting of 16 base elements, e_j, and an expression of the form

$$z = \sum_{j=0}^{15} e_j x_j, \quad x_j \ \ (j = 0, 1, \ldots, 15) \in \mathbb{R}. \tag{1}$$

The Table 1 below displays the rules of the 16 bases that comprise a sedenion.

Table 1. Rules of 16 base elements, e_j.

$\times$	$\boldsymbol{e_0}$	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	$\boldsymbol{e_4}$	$\boldsymbol{e_5}$	$\boldsymbol{e_6}$	$\boldsymbol{e_7}$	$\boldsymbol{e_8}$	$\boldsymbol{e_9}$	$\boldsymbol{e_{10}}$	$\boldsymbol{e_{11}}$	$\boldsymbol{e_{12}}$	$\boldsymbol{e_{13}}$	$\boldsymbol{e_{14}}$	$\boldsymbol{e_{15}}$
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$	e_9	$-e_8$	e_{11}	$-e_{10}$	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	-1	e_1	e_6	$-e_7$	$-e_4$	e_5	e_{10}	$-e_{11}$	$-e_8$	e_9	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}
e_3	e_3	e_2	$-e_1$	-1	e_7	e_6	$-e_5$	$-e_4$	e_{11}	e_{10}	$-e_9$	$-e_8$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	e_7	$-e_6$	$-e_1$	-1	e_3	$-e_2$	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	$-e_7$	e_4	e_5	$-e_2$	$-e_3$	-1	e_1	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	e_6	$-e_5$	e_4	$-e_3$	e_2	$-e_1$	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$

Table 1. *Cont.*

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	e_{11}	$-e_{10}$	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	e_3	$-e_2$	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	$-e_{11}$	e_8	e_9	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	$-e_3$	-1	e_1	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	e_{10}	$-e_9$	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	e_2	$-e_1$	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	$-e_{13}$	$-e_{14}$	$-e_{15}$	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	e_1	e_2	e_3
e_{13}	e_{13}	e_{12}	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	$-e_1$	-1	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	e_{12}	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	$-e_2$	$-e_3$	-1	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	e_{12}	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	$-e_3$	$-e_2$	$-e_1$	-1

The field of complex numbers, $\mathbb{C}$, is equipped with an identity element, e_0, and an imaginary unit, $e_1 = \sqrt{-1}$. A sedenion, z, is represented by the form $z = \zeta_0 + e_8\overline{\zeta_1}$, where $\zeta_0 := \sum_{j=0}^{7} e_j x_j$ and $\zeta_1 := \sum_{j=0}^{7} e_j x_{j+8}$ are octonions, $\mathbb{O}$. The set $\mathbb{O} \times \mathbb{O}$, where $\mathbb{O}$ denotes the octonions, is non-commutative. It can be identified with $\mathbb{O}^2$ and is defined as $\{z = e_0\zeta_0 + e_8\zeta_1 | \zeta_j \in \mathbb{O}, j = 0, 1\}$.

We give the non-commutative multiplication, $\widetilde{\times}$, of two sedenions, $z = \zeta_0 + e_8\overline{\zeta_1}$ and $w = \eta_0 + e_8\overline{\eta_1}$, denoted by

$$z\widetilde{\times}w = \frac{1}{2}(\zeta_0\eta_0 - \zeta_1\overline{\eta_1} + \eta_0\zeta_0 - \eta_1\overline{\zeta_1}) + \frac{1}{2}e_8(\overline{\zeta_0}\overline{\eta_1} + \overline{\zeta_1}\eta_0 - \overline{\eta_0}\overline{\zeta_1} - \overline{\eta_1}\zeta_0)$$

and

$$\begin{aligned} w\widetilde{\times}z &= \frac{1}{2}(\zeta_0\eta_0 - \zeta_1\overline{\eta_1} + \eta_0\zeta_0 - \eta_1\overline{\zeta_1}) + \frac{1}{2}e_8(-\overline{\zeta_0}\overline{\eta_1} - \overline{\zeta_1}\eta_0 + \overline{\eta_0}\overline{\zeta_1} + \overline{\eta_1}\zeta_0) \\ &= \frac{1}{2}(\zeta_0\eta_0 - \zeta_1\overline{\eta_1} + \eta_0\zeta_0 - \eta_1\overline{\zeta_1}) - \frac{1}{2}e_8(\overline{\zeta_0}\overline{\eta_1} + \overline{\zeta_1}\eta_0 - \overline{\eta_0}\overline{\zeta_1} - \overline{\eta_1}\zeta_0), \end{aligned}$$

where

$$\overline{\zeta_j} = \sum_{i=0}^{7} \overline{e_j} x_j \quad \text{and} \quad \overline{\eta_j} = \sum_{i=0}^{7} \overline{e_j} y_j \quad (j = 0, 1).$$

The process for calculating variables in the real number system is detailed in Appendix A. The two equations, $z\widetilde{\times}w$ and $w\widetilde{\times}z$, differ only in the sign of the basis, e_8.

The absolute value, $|z|$, and sedenionic conjugate number, z^*, of $z = \zeta_0 + e_8\zeta_1$ are given by

$$|z| = \sqrt{z\widetilde{\times}z^*} = \sqrt{z^*\widetilde{\times}z} = \sqrt{\sum_{j=0}^{15} x_j^2} \quad \text{and} \quad z^* = \overline{\zeta_0} - e_8\overline{\zeta_1}.$$

3. Regularity of a Function with Values in $\mathbb{O} \times \mathbb{O}$

Let Ω be an open subset of $\mathbb{O} \times \mathbb{O}$. We can define the function $f(z)$ in the domain Ω as $f : \Omega \to \mathbb{O}^2$, such that it satisfies the following equation:

$$z = (\zeta_0, \zeta_1) \in \Omega \mapsto f(z) = f(\zeta_0, \zeta_1) = f_0(\zeta_0, \zeta_1) + e_8\overline{f_1}(\zeta_0, \zeta_1) \in \mathbb{O}^2,$$

where $f_0 = \sum_{j=0}^{7} e_j u_j$, $f_1 = \sum_{j=0}^{7} \overline{e_j} u_{j+8}$ and u_j $(j = 0, 1, \ldots, 15)$ are real-valued functions.

We consider the differential operators as follows:

$$\begin{aligned} D_1 &= \frac{\partial}{\partial\zeta_0} - e_8\frac{\partial}{\partial\overline{\zeta_1}}, \quad D_1^* = \frac{\partial}{\partial\overline{\zeta_0}} + e_8\frac{\partial}{\partial\overline{\zeta_1}} \\ D_2 &= \frac{\partial}{\partial\zeta_0} - \frac{\partial}{\partial\overline{\zeta_1}}e_8, \quad D_2^* = \frac{\partial}{\partial\overline{\zeta_0}} + \frac{\partial}{\partial\overline{\zeta_1}}e_8, \end{aligned}$$

where $\frac{\partial}{\partial \zeta_0} = \sum_{j=0}^{7} \overline{e_j} \frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial \zeta_1} = \sum_{j=0}^{7} \overline{e_j} \frac{\partial}{\partial x_{j+8}}$.

We have the Laplacian of $\mathbb{O}^2$:

$$D_k \widetilde{\times} D_k^* = \frac{\partial^2}{\partial \zeta_0 \partial \overline{\zeta_0}} + \frac{\partial^2}{\partial \zeta_1 \partial \overline{\zeta_1}} = \sum_{j=0}^{15} \frac{\partial^2}{\partial x_j^2},$$

where $k = 1,2$.

We have derived the following equations from the operators mentioned above:

$$D_1 \widetilde{\times} f = \left(\frac{\partial f_0}{\partial \zeta_0} + \frac{\partial \overline{f_1}}{\partial \zeta_1} + f_0 \frac{\partial}{\partial \zeta_0} + f_1 \frac{\partial}{\overline{\partial \overline{\zeta_1}}} \right) + e_8 \left(\frac{\partial \overline{f_1}}{\overline{\partial \overline{\zeta_0}}} - \frac{\partial f_0}{\overline{\partial \overline{\zeta_1}}} + \overline{f_0} \frac{\partial}{\overline{\partial \overline{\zeta_1}}} - \overline{f_1} \frac{\partial}{\partial \zeta_0} \right), \tag{2}$$

$$D_1^* \widetilde{\times} f = \left(\frac{\partial f_0}{\overline{\partial \overline{\zeta_0}}} - \frac{\partial \overline{f_1}}{\partial \zeta_1} + f_0 \frac{\partial}{\overline{\partial \overline{\zeta_0}}} - f_1 \frac{\partial}{\overline{\partial \overline{\zeta_1}}} \right) + e_8 \left(\frac{\partial \overline{f_1}}{\partial \zeta_0} + \frac{\partial f_0}{\overline{\partial \overline{\zeta_1}}} - \overline{f_0} \frac{\partial}{\overline{\partial \overline{\zeta_1}}} - \overline{f_1} \frac{\partial}{\overline{\partial \overline{\zeta_0}}} \right), \tag{3}$$

$$D_2 \widetilde{\times} f = \left(\frac{\partial f_0}{\partial \zeta_0} + \frac{\partial \overline{f_1}}{\overline{\partial \overline{\zeta_1}}} + f_0 \frac{\partial}{\partial \zeta_0} + f_1 \frac{\partial}{\partial \zeta_1} \right) + e_8 \left(\frac{\partial \overline{f_1}}{\overline{\partial \overline{\zeta_0}}} - \frac{\partial f_0}{\partial \zeta_1} + \overline{f_0} \frac{\partial}{\partial \zeta_1} - \overline{f_1} \frac{\partial}{\partial \zeta_0} \right) \tag{4}$$

and

$$D_2^* \widetilde{\times} f = \left(\frac{\partial f_0}{\overline{\partial \overline{\zeta_0}}} - \frac{\partial \overline{f_1}}{\overline{\partial \overline{\zeta_1}}} + f_0 \frac{\partial}{\overline{\partial \overline{\zeta_0}}} - f_1 \frac{\partial}{\partial \zeta_1} \right) + e_8 \left(\frac{\partial \overline{f_1}}{\partial \zeta_0} + \frac{\partial f_0}{\partial \zeta_1} - \overline{f_0} \frac{\partial}{\partial \zeta_1} - \overline{f_1} \frac{\partial}{\overline{\partial \overline{\zeta_0}}} \right). \tag{5}$$

In order to understand how each term in Equations (2)–(5) is calculated, we have provided a detailed calculation process in Appendix B using the real term system. The regularity of functions with values in $\mathbb{O} \times \mathbb{O}$ is defined by the results of Equations (2)–(5). The following definitions and theorems are based on the works of Li et al. [14] and Ludkovski [15].

Definition 1. *Let Ω be a domain in $\mathbb{O} \times \mathbb{O}$. A function $f(z) = f_0(\zeta_0, \zeta_1) + e_8 \overline{f_1}(\zeta_0, \zeta_1)$ is said to be $L_k(R_k)$-regular in Ω if the following two conditions are satisfied:*

(i) f_0 *and* f_1 *are differential functions in* Ω*,*
(ii) $D_k^* \widetilde{\times} f(z) = 0 \quad (f(z) \widetilde{\times} D_k^* = 0)$ *in* Ω*, where* $k = 1,2$.

The equations (ii) mentioned in Definition 1 are equivalent to the following equations: for $k = 1$,

$$\frac{\partial f_0}{\overline{\partial \overline{\zeta_0}}} + f_0 \frac{\partial}{\overline{\partial \overline{\zeta_0}}} = \frac{\partial \overline{f_1}}{\partial \zeta_1} + f_1 \frac{\partial}{\overline{\partial \overline{\zeta_1}}}, \quad \frac{\partial \overline{f_1}}{\partial \zeta_0} - \overline{f_1} \frac{\partial}{\overline{\partial \overline{\zeta_0}}} = -\frac{\partial f_0}{\overline{\partial \overline{\zeta_1}}} + \overline{f_0} \frac{\partial}{\overline{\partial \overline{\zeta_1}}}, \tag{6}$$

and for $k = 2$,

$$\frac{\partial f_0}{\overline{\partial \overline{\zeta_0}}} + f_0 \frac{\partial}{\overline{\partial \overline{\zeta_0}}} = \frac{\partial \overline{f_1}}{\overline{\partial \overline{\zeta_1}}} + f_1 \frac{\partial}{\partial \zeta_1}, \quad \frac{\partial \overline{f_1}}{\partial \zeta_0} - \overline{f_1} \frac{\partial}{\overline{\partial \overline{\zeta_0}}} = -\frac{\partial f_0}{\partial \zeta_1} + \overline{f_0} \frac{\partial}{\partial \zeta_1}. \tag{7}$$

Moreover, $f(z) \widetilde{\times} D_k^* = 0$ if and only if $D_k^* \overline{\times} f(z) = 0$ $(k = 1,2)$.

As referred to by Kauhanen and Orelma [16] and Vinogradov [17], the equations in (6) and (7) are the corresponding Cauchy–Riemann for the $f(z)$ system in $\mathbb{T}$. Let $f(z)$ be a $L_k(R_k)$ $(k = 1,2)$-regular function defined in Ω. The derivative $f'(z)$ of $f(z)$ is defined by $f'(z) = D \widetilde{\times} f(z)$.

According to Kauhanen and Orelma [16] and Vinogradov [17], Equations (6) and (7) represent the Cauchy–Riemann system for the function, $f(z)$, in $\mathbb{T}$. For a $L_k(R_k)$ $(k=1,2)$-regular function $f(z)$ defined in Ω, we can define its derivative $f'(z)$ as $f'(z) = D\widetilde{\times} f(z)$.

Theorem 1. *Let Ω be a domain in $\mathbb{O} \times \mathbb{O}$ and $f(z)$ be a $L_k(R_k)$-regular function defined on Ω, where $(k=1,2)$. Then,*

$$f'(z) = \left(\frac{\partial}{\partial \zeta_0} + \frac{\partial}{\partial \overline{\zeta_0}}\right)\widetilde{\times} f = 2\frac{\partial}{\partial x_0}\widetilde{\times} f.$$

Proof. Since the function $f(z)$ is $L_k(R_k)$-regular in Ω, we have

$$\begin{aligned} D_1\widetilde{\times} f &= \left(\frac{\partial f_0}{\partial \zeta_0} + \frac{\partial \overline{f_1}}{\partial \zeta_1} + f_0\frac{\partial}{\partial \zeta_0} + f_1\frac{\partial}{\partial \overline{\zeta_1}}\right) + e_8\left(\frac{\partial \overline{f_1}}{\partial \overline{\zeta_0}} - \frac{\partial f_0}{\partial \overline{\zeta_1}} + \overline{f_0}\frac{\partial}{\partial \overline{\zeta_1}} - \overline{f_1}\frac{\partial}{\partial \zeta_0}\right) \\ &= \left(\frac{\partial}{\partial \zeta_0} + \frac{\partial}{\partial \overline{\zeta_0}}\right)f_0 + f_0\left(\frac{\partial}{\partial \zeta_0} + \frac{\partial}{\partial \overline{\zeta_0}}\right) + e_8\left\{\left(\frac{\partial}{\partial \zeta_0} + \frac{\partial}{\partial \overline{\zeta_0}}\right)\overline{f_1} - \overline{f_1}\left(\frac{\partial}{\partial \zeta_0} + \frac{\partial}{\partial \overline{\zeta_0}}\right)\right\} \\ &= \left(\frac{\partial}{\partial \zeta_0} + \frac{\partial}{\partial \overline{\zeta_0}}\right)\widetilde{\times} f. \end{aligned}$$

Similarly, we obtain the equation $D_2\widetilde{\times} f = \left(\frac{\partial}{\partial \zeta_0} + \frac{\partial}{\partial \overline{\zeta_0}}\right)\widetilde{\times} f.$ □

Theorem 2. *Let a function, $f(z)$, be $L_k(R_k)$-regular in a domain G of $\mathbb{O} \times \mathbb{O}$ and let*

$$\begin{aligned} \tau \;=\; & dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_2} \wedge d\overline{z_3} \wedge d\overline{z_4} \\ & - dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_1} \wedge d\overline{z_3} \wedge d\overline{z_4} e_2 \\ & + dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{z_4} e_4 \\ & - dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{z_3} e_6. \end{aligned}$$

Then, for any domain $\Omega \subset G$ with smooth boundary $b\Omega$,

$$\int_{b\Omega} \tau g = 0.$$

Proof. Let

$$\begin{aligned} \tau_{(1)} &= dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_2} \wedge d\overline{z_3} \wedge d\overline{z_4}, \\ \tau_{(2)} &= dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_1} \wedge d\overline{z_3} \wedge d\overline{z_4}, \\ \tau_{(3)} &= dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{z_4}, \\ \tau_{(4)} &= dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{z_3}. \end{aligned}$$

Then,

$$\begin{aligned} \tau g &= (\tau_{(1)} - \tau_{(2)}e_2 + \tau_{(3)}e_4 - \tau_{(4)}e_6)(g_1 + g_2e_2 + g_3e_4 + g_4e_6) \\ &= (g_1\tau_{(1)} - g_2\tau_{(2)} + g_3\tau_{(3)} - g_4\tau_{(4)}) + (g_2\tau_{(1)} - g_1\tau_{(2)} + g_4\tau_{(3)} - g_3\tau_{(4)})e_2 \\ &\quad + (g_3\tau_{(1)} - g_4\tau_{(2)} + g_1\tau_{(3)} - g_2\tau_{(4)})e_4 + (g_4\tau_{(1)} - g_3\tau_{(2)} + g_2\tau_{(3)} - g_1\tau_{(4)})e_6. \end{aligned}$$

Hence,

$$\begin{aligned} d(\tau g) \;=\; & \left(\frac{\partial g_1}{\partial \overline{z_1}} + \frac{\partial g_2}{\partial \overline{z_2}} + \frac{\partial g_3}{\partial \overline{z_3}} + \frac{\partial g_4}{\partial \overline{z_4}}\right)dV + \left(\frac{\partial g_2}{\partial \overline{z_1}} + \frac{\partial g_1}{\partial \overline{z_2}} + \frac{\partial g_4}{\partial \overline{z_3}} + \frac{\partial g_3}{\partial \overline{z_4}}\right)dVe_2 \\ & + \left(\frac{\partial g_3}{\partial \overline{z_1}} + \frac{\partial g_4}{\partial \overline{z_2}} + \frac{\partial g_1}{\partial \overline{z_3}} + \frac{\partial g_2}{\partial \overline{z_4}}\right)dVe_4 + \left(\frac{\partial g_4}{\partial \overline{z_1}} + \frac{\partial g_3}{\partial \overline{z_2}} + \frac{\partial g_2}{\partial \overline{z_3}} + \frac{\partial g_1}{\partial \overline{z_4}}\right)dVe_6, \end{aligned}$$

where $dV = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{z_3} \wedge d\overline{z_4}$, and by Equations (6) and (7) we obtain $d(\tau g) = 0$. By Stoke's theorem, we have

$$\int_{b\Omega} \tau g = \int_{\Omega} d(\tau g) = 0.$$

□

Theorem 3. *Let Ω be a domain of holomorphy, $\mathbb{C}(\mathbb{C})$, with respect to the complex variables z_1, z_2, z_3, z_4. For any complex valued function $g_i(z) (i = 1, 2, 3, 4)$ of class $\mathcal{C}^2$ on Ω, we can find another complex valued function $g_i(z) (i = 1, 2, 3, 4)$ of class $\mathcal{C}^2$ on Ω, such that $g(z) = g_1(z) + g_2(z)e_2 + g_3(z)e_4 + g_4(z)e_6$ is a O-regular function on Ω.*

Proof. If $g_1(z)$ is a complex-valued function of class $\mathcal{C}^2$ on Ω, then there exists a complex-valued function $g_2(z)$ of class $\mathcal{C}^2$ on Ω. We consider the 1-form and the differential operator on Ω:

$$\begin{aligned} \psi \ := \ & -(\frac{\partial g_1}{\partial \overline{z_2}} + \frac{\partial g_4}{\partial \overline{z_3}} + \frac{\partial g_3}{\partial \overline{z_4}})d\overline{z_1} - (\frac{\partial g_1}{\partial \overline{z_1}} + \frac{\partial g_3}{\partial \overline{z_3}} + \frac{\partial g_4}{\partial \overline{z_4}})d\overline{z_2} \\ & -(\frac{\partial g_4}{\partial \overline{z_1}} + \frac{\partial g_3}{\partial \overline{z_2}} + \frac{\partial g_1}{\partial \overline{z_4}})d\overline{z_3} - (\frac{\partial g_3}{\partial \overline{z_1}} + \frac{\partial g_4}{\partial \overline{z_2}} + \frac{\partial g_1}{\partial \overline{z_3}})d\overline{z_4}. \end{aligned}$$

Then,we operate the operator, $\overline{\partial}$, from the left-hand side for the 1-form ψ on Ω:

$$\begin{aligned} \overline{\partial}\psi \ = \ & (-\frac{\partial^2 g_1}{\partial \overline{z_1} \partial \overline{z_1}} - \frac{\partial^2 g_3}{\partial \overline{z_1} \partial \overline{z_3}} - \frac{\partial^2 g_4}{\partial \overline{z_1} \partial \overline{z_4}} + \frac{\partial^2 g_1}{\partial \overline{z_2} \partial \overline{z_2}} + \frac{\partial^2 g_4}{\partial \overline{z_2} \partial \overline{z_3}} + \frac{\partial^2 g_3}{\partial \overline{z_2} \partial \overline{z_4}})d\overline{z_1} \wedge d\overline{z_2} \\ & (-\frac{\partial^2 g_4}{\partial \overline{z_1} \partial \overline{z_1}} - \frac{\partial^2 g_3}{\partial \overline{z_1} \partial \overline{z_2}} - \frac{\partial^2 g_1}{\partial \overline{z_1} \partial \overline{z_4}} + \frac{\partial^2 g_1}{\partial \overline{z_3} \partial \overline{z_2}} + \frac{\partial^2 g_4}{\partial \overline{z_3} \partial \overline{z_3}} + \frac{\partial^2 g_3}{\partial \overline{z_3} \partial \overline{z_4}})d\overline{z_1} \wedge d\overline{z_3} \\ & (-\frac{\partial^2 g_3}{\partial \overline{z_1} \partial \overline{z_1}} - \frac{\partial^2 g_4}{\partial \overline{z_1} \partial \overline{z_2}} - \frac{\partial^2 g_1}{\partial \overline{z_1} \partial \overline{z_3}} + \frac{\partial^2 g_1}{\partial \overline{z_4} \partial \overline{z_2}} + \frac{\partial^2 g_4}{\partial \overline{z_4} \partial \overline{z_3}} + \frac{\partial^2 g_3}{\partial \overline{z_4} \partial \overline{z_4}})d\overline{z_1} \wedge d\overline{z_4} \\ & (-\frac{\partial^2 g_4}{\partial \overline{z_2} \partial \overline{z_1}} - \frac{\partial^2 g_3}{\partial \overline{z_2} \partial \overline{z_2}} - \frac{\partial^2 g_1}{\partial \overline{z_2} \partial \overline{z_4}} + \frac{\partial^2 g_1}{\partial \overline{z_3} \partial \overline{z_1}} + \frac{\partial^2 g_3}{\partial \overline{z_3} \partial \overline{z_3}} + \frac{\partial^2 g_4}{\partial \overline{z_3} \partial \overline{z_4}})d\overline{z_2} \wedge d\overline{z_3} \\ & (-\frac{\partial^2 g_3}{\partial \overline{z_2} \partial \overline{z_1}} - \frac{\partial^2 g_4}{\partial \overline{z_2} \partial \overline{z_2}} - \frac{\partial^2 g_1}{\partial \overline{z_2} \partial \overline{z_3}} + \frac{\partial^2 g_1}{\partial \overline{z_4} \partial \overline{z_1}} + \frac{\partial^2 g_3}{\partial \overline{z_4} \partial \overline{z_3}} + \frac{\partial^2 g_4}{\partial \overline{z_4} \partial \overline{z_4}})d\overline{z_2} \wedge d\overline{z_4} \\ & (-\frac{\partial^2 g_3}{\partial \overline{z_3} \partial \overline{z_1}} - \frac{\partial^2 g_4}{\partial \overline{z_3} \partial \overline{z_2}} - \frac{\partial^2 g_1}{\partial \overline{z_3} \partial \overline{z_3}} + \frac{\partial^2 g_4}{\partial \overline{z_4} \partial \overline{z_1}} + \frac{\partial^2 g_3}{\partial \overline{z_4} \partial \overline{z_2}} + \frac{\partial^2 g_1}{\partial \overline{z_4} \partial \overline{z_4}})d\overline{z_3} \wedge d\overline{z_4}. \end{aligned}$$

Based on Equations (6) and (7), all coefficients turn out to be zero. According to Hörmander [18], the $\overline{\partial}$-closed form ψ of z_1, z_2, z_3, and z_4 is a $\overline{\partial}$-exact form on Ω. Krantz [19] states that since Ω is a domain of holomorphy, there exists a complex-valued function $g_i(z)$ (where $i = 1, 2, 3, 4$) of class $\mathcal{C}^\infty$ on Ω, with a $\overline{\partial}$-closed form $\psi = \overline{\partial} g_2(z)$ on Ω. The form ψ is of z_1, z_2, z_3, and z_4 and is a $\overline{\partial}$-exact $(0, 1)$-form on Ω. Moreover, $g(z)$ is a O-regular function on Ω. Using a similar approach, we can derive the desired outcome. □

4. Conclusions

The sedenion is a number system consisting of 16 real numbers and their bases. It can be expressed in the dual structure of octonions. By using the properties and characteristics of octonions, we can investigate the regular function of a sedenion and the Cauchy–Riemann equations derived from regularity. To understand the structure of a sedenion extended from octonions, we can express it with two octonions. This helps us present a regular function defined in the variable representing sedenion. We can also specify the Cauchy–Riemann equations on sedenions corresponding to the structure of the two octonions. By constructing Cauchy–Riemann equations, it is easy to determine the regular function on a sedenion. We can also identify the characteristics of the regular function on a sedenion. By interpreting the expansion of the number system using the structure of the lower number system, we can utilize the analytical properties and structure of the lower number system. Based on the properties and structure of octonions, the regularity of a sedenion is defined. This helps us derive the Cauchy–Riemann equation forms and applications. We can investigate the properties of a regular function and their relationship with Cauchy–Riemann equations.

Operators in the complex sedenion number system are newly defined and used in the corresponding number system. Although the operation is difficult to utilize in existing lower number systems, the convenience of complex structures and the ability to quote the algebraic results of lower number systems, especially quaternions and octonions, enables the application of additional function theory and research into new proposals. Through the attempts and results made in this paper, the use of a number system citing complex number structures enables the formation of a higher-level number system. Also, based on this, the following recent literature related to the use of lower number systems that deal with complex number structures can be introduced.

In the field of signal processing, a theory proposed by Bhat et al. [20] has introduced a new tool called the Wigner–Ville distribution (WVD) associated with quaternion offset linear canonical transform (QOLCT). The main contribution of this work is the introduction of WVD–QOLCT and an ambiguity function (AF) associated with QOLCT (AF–QOLCT). Firstly, it proposed a definition of WVD–QOLCT and derived several important properties such as dilation, nonlinearity, and boundedness. Secondly, they derived the AF for the proposed transform and studied a number of important properties, including the reconstruction formula associated with the AF. This work is an emerging tool in the field of signal processing and can potentially contribute towards solving various signal processing-related challenges. Oubba [21] has presented a comprehensive proof that establishes the validity of any local and 2-local derivation of the generalized quaternion algebra $\mathcal{H}_{\alpha,\beta}$ as a derivation. This proof will entail a systematic and rigorous analysis of the algebraic structure of $\mathcal{H}_{\alpha,\beta}$, as well as a thorough examination of its various properties and characteristics. Furthermore, this paper will also focus on determining the biderivations, commuting linear maps, and centroid of $\mathcal{H}_{\alpha,\beta}$ over the real numbers. This will require an in-depth investigation of the various linear maps and their properties, as well as an analysis of the algebraic operations that govern their behavior in relation to $\mathcal{H}_{\alpha,\beta}$. Through this analysis, we aim to provide a comprehensive understanding of the algebraic structure of $\mathcal{H}_{\alpha,\beta}$ and its various properties. Achak et al. [22] have discussed some estimates that have various applications in interpolation theory. In particular, recent issues in image processing and singular integral operators require the computation of appropriate estimates. Abilov et al. [23] have proved two useful estimates for the Fourier transform in the space of square integral multivariable functions on certain classes of functions characterized by the generalized continuity modulus. These estimates are proven for only two variables, using a translation operator. The purpose of this paper is to study these estimates for measurable sets from the complex domain to a hypercomplex domain by using quaternion algebras, associated with the quaternion linear canonical transform, constructed by the generalized Steklov function.

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Conflicts of Interest: The author declares no conflicts of interest.

Appendix A

The specific calculation process in the real number system for variables is developed in

$$
\begin{aligned}
\overline{\zeta_0\zeta_1} = {} & x_0(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& - e_1x_1(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& - e_2x_2(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& - e_3x_3(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& - e_4x_4(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& - e_5x_5(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& - e_6x_6(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& - e_7x_7(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}),
\end{aligned}
$$

$$
\begin{aligned}
\overline{\zeta_0\zeta_1} = {} & (x_0x_8 - e_1x_0x_9 - e_2x_0x_{10} - e_3x_0x_{11} - e_4x_0x_{12} - e_5x_0x_{13} - e_6x_0x_{14} - e_7x_0x_{15}) \\
& + (-e_1x_1x_8 - x_1x_9 + e_3x_1x_{10} - e_2x_1x_{11} + e_5x_1x_{12} - e_4x_1x_{13} - e_7x_1x_{14} + e_6x_1x_{15}) \\
& + (-e_2x_2x_8 - e_3x_2x_9 - x_2x_{10} + e_1x_2x_{11} + e_6x_2x_{12} + e_7x_2x_{13} - e_4x_2x_{14} - e_5x_2x_{15}) \\
& + (-e_3x_3x_8 + e_2x_3x_9 - e_1x_3x_{10} - x_3x_{11} + e_7x_3x_{12} - e_6x_3x_{13} + e_5x_3x_{14} - e_4x_3x_{15}) \\
& + (-e_4x_4x_8 - e_5x_4x_9 - e_6x_4x_{10} - e_7x_4x_{11} - x_4x_{12} + e_1x_4x_{13} + e_2x_4x_{14} + e_3x_4x_{15}) \\
& + (-e_5x_5x_8 + e_4x_5x_9 - e_7x_5x_{10} + e_6x_5x_{11} - e_1x_5x_{12} - x_5x_{13} - e_3x_5x_{14} + e_2x_5x_{15}) \\
& + (-e_6x_6x_8 + e_7x_6x_9 + e_4x_6x_{10} - e_5x_6x_{11} - e_2x_6x_{12} + e_3x_6x_{13} - x_6x_{14} - e_1x_6x_{15}) \\
& + (-e_7x_7x_8 - e_6x_7x_9 + e_5x_7x_{10} + e_4x_7x_{11} - e_3x_7x_{12} - e_2x_7x_{13} + e_1x_7x_{14} - x_7x_{15}),
\end{aligned}
$$

$$
\begin{aligned}
\overline{\zeta_1\zeta_0} = {} & x_8(x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7) \\
& - e_1x_9(x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7) \\
& - e_2x_{10}(x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7) \\
& - e_3x_{11}(x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7) \\
& - e_4x_{12}(x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7) \\
& - e_5x_{13}(x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7) \\
& - e_6x_{14}(x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7) \\
& - e_7x_{15}(x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7),
\end{aligned}
$$

$$
\begin{aligned}
\overline{\zeta_1\zeta_0} = {} & (x_8x_0 - e_1x_8x_1 - e_2x_8x_2 - e_3x_8x_3 - e_4x_8x_4 - e_5x_8x_5 - e_6x_8x_6 - e_7x_8x_7) \\
& + (-e_1x_9x_0 - x_9x_1 + e_3x_9x_2 - e_2x_9x_3 + e_5x_9x_4 - e_4x_9x_5 - e_7x_9x_6 + e_6x_9x_7) \\
& + (-e_2x_{10}x_0 - e_3x_{10}x_1 - x_{10}x_2 + e_1x_{10}x_3 + e_6x_{10}x_4 + e_7x_{10}x_5 - e_4x_{10}x_6 - e_5x_{10}x_7) \\
& + (-e_3x_{11}x_0 + e_2x_{11}x_1 - e_1x_{11}x_2 - x_{11}x_3 + e_7x_{11}x_4 - e_6x_{11}x_5 + e_5x_{11}x_6 - e_4x_{11}x_7) \\
& + (-e_4x_{12}x_0 - e_5x_{12}x_1 - e_6x_{12}x_2 - e_7x_{12}x_3 - x_{12}x_4 + e_1x_{12}x_5 + e_2x_{12}x_6 + e_3x_{12}x_7) \\
& + (-e_5x_{13}x_0 + e_4x_{13}x_1 - e_7x_{13}x_2 + e_6x_{13}x_3 - e_1x_{13}x_4 - x_{13}x_5 - e_3x_{13}x_6 + e_2x_{13}x_7) \\
& + (-e_6x_{14}x_0 + e_7x_{14}x_1 + e_4x_{14}x_2 - e_5x_{14}x_3 - e_2x_{14}x_4 + e_3x_{14}x_5 - x_{14}x_6 - e_1x_{14}x_7) \\
& + (-e_7x_{15}x_0 - e_6x_{15}x_1 + e_5x_{15}x_2 + e_4x_{15}x_3 - e_3x_{15}x_4 - e_2x_{15}x_5 + e_1x_{15}x_6 - x_{15}x_7),
\end{aligned}
$$

$$
\begin{aligned}
\zeta_0\overline{\zeta_1} = {} & x_0(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& + e_1x_1(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& + e_2x_2(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& + e_3x_3(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& + e_4x_4(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& + e_5x_5(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& + e_6x_6(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}) \\
& + e_7x_7(x_8 - e_1x_9 - e_2x_{10} - e_3x_{11} - e_4x_{12} - e_5x_{13} - e_6x_{14} - e_7x_{15}),
\end{aligned}
$$

$$\begin{aligned}
\zeta_0\overline{\zeta_1} = {} & (x_0x_8 - e_1x_0x_9 - e_2x_0x_{10} - e_3x_0x_{11} - e_4x_0x_{12} - e_5x_0x_{13} - e_6x_0x_{14} - e_7x_0x_{15}) \\
& + (e_1x_1x_8 + x_1x_9 - e_3x_1x_{10} + e_2x_1x_{11} - e_5x_1x_{12} + e_4x_1x_{13} + e_7x_1x_{14} - e_6x_1x_{15}) \\
& + (e_2x_2x_8 + e_3x_2x_9 + x_2x_{10} - e_1x_2x_{11} - e_6x_2x_{12} - e_7x_2x_{13} + e_4x_2x_{14} + e_5x_2x_{15}) \\
& + (e_3x_3x_8 - e_2x_3x_9 + e_1x_3x_{10} + x_3x_{11} - e_7x_3x_{12} + e_6x_3x_{13} - e_5x_3x_{14} + e_4x_3x_{15}) \\
& + (e_4x_4x_8 + e_5x_4x_9 + e_6x_4x_{10} + e_7x_4x_{11} + x_4x_{12} - e_1x_4x_{13} - e_2x_4x_{14} - e_3x_4x_{15}) \\
& + (e_5x_5x_8 - e_4x_5x_9 + e_7x_5x_{10} - e_6x_5x_{11} + e_1x_5x_{12} + x_5x_{13} + e_3x_5x_{14} - e_2x_5x_{15}) \\
& + (e_6x_6x_8 - e_7x_6x_9 - e_4x_6x_{10} + e_5x_6x_{11} + e_2x_6x_{12} - e_3x_6x_{13} + x_6x_{14} + e_1x_6x_{15}) \\
& + (e_7x_7x_8 + e_6x_7x_9 - e_5x_7x_{10} - e_4x_7x_{11} + e_3x_7x_{12} + e_2x_7x_{13} - e_1x_7x_{14} + x_7x_{15}),
\end{aligned}$$

$$\begin{aligned}
\overline{\zeta_1}\zeta_0 = {} & x_8(x_0 + e_1x_1 + e_2x_2 - e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7) \\
& - e_1x_9(x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7) \\
& - e_2x_{10}(x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7) \\
& - e_3x_{11}(x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7) \\
& - e_4x_{12}(x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7) \\
& - e_5x_{13}(x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7) \\
& - e_6x_{14}(x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7) \\
& - e_7x_{15}(x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7),
\end{aligned}$$

and

$$\begin{aligned}
\overline{\zeta_1}\zeta_0 = {} & (x_8x_0 + e_1x_8x_1 + e_2x_8x_2 + e_3x_8x_3 + e_4x_8x_4 + e_5x_8x_5 + e_6x_8x_6 + e_7x_8x_7) \\
& + (-e_1x_9x_0 + x_9x_1 - e_3x_9x_2 + e_2x_9x_3 - e_5x_9x_4 + e_4x_9x_5 + e_7x_9x_6 - e_6x_9x_7) \\
& + (-e_2x_{10}x_0 + e_3x_{10}x_1 + x_{10}x_2 - e_1x_{10}x_3 - e_6x_{10}x_4 - e_7x_{10}x_5 + e_4x_{10}x_6 + e_5x_{10}x_7) \\
& + (-e_3x_{11}x_0 - e_2x_{11}x_1 + e_1x_{11}x_2 + x_{11}x_3 - e_7x_{11}x_4 + e_6x_{11}x_5 - e_5x_{11}x_6 + e_4x_{11}x_7) \\
& + (-e_4x_{12}x_0 + e_5x_{12}x_1 + e_6x_{12}x_2 + e_7x_{12}x_3 + x_{12}x_4 - e_1x_{12}x_5 - e_2x_{12}x_6 - e_3x_{12}x_7) \\
& + (-e_5x_{13}x_0 - e_4x_{13}x_1 + e_7x_{13}x_2 - e_6x_{13}x_3 + e_1x_{13}x_4 + x_{13}x_5 + e_3x_{13}x_6 - e_2x_{13}x_7) \\
& + (-e_6x_{14}x_0 - e_7x_{14}x_1 - e_4x_{14}x_2 + e_5x_{14}x_3 + e_2x_{14}x_4 - e_3x_{14}x_5 + x_{14}x_6 + e_1x_{14}x_7) \\
& + (-e_7x_{15}x_0 + e_6x_{15}x_1 - e_5x_{15}x_2 - e_4x_{15}x_3 + e_3x_{15}x_4 + e_2x_{15}x_5 - e_1x_{15}x_6 + x_{15}x_7).
\end{aligned}$$

Appendix B

In order to examine the specific calculation process for each term expressed in Equations (2)–(5), the calculation is developed in the real term system in

$$\begin{aligned}
\frac{\partial}{\partial\zeta_0}f_0 = {} & \frac{\partial}{\partial x_0}(u_0 + e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_5 + e_6u_6 + e_7u_7) \\
& - e_1\frac{\partial}{\partial x_1}(u_0 + e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_5 + e_6u_6 + e_7u_7) \\
& - e_2\frac{\partial}{\partial x_2}(u_0 + e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_5 + e_6u_6 + e_7u_7) \\
& - e_3\frac{\partial}{\partial x_3}(u_0 + e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_5 + e_6u_6 + e_7u_7) \\
& - e_4\frac{\partial}{\partial x_4}(u_0 + e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_5 + e_6u_6 + e_7u_7) \\
& - e_5\frac{\partial}{\partial x_5}(u_0 + e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_5 + e_6u_6 + e_7u_7) \\
& - e_6\frac{\partial}{\partial x_6}(u_0 + e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_5 + e_6u_6 + e_7u_7) \\
& - e_7\frac{\partial}{\partial x_7}(u_0 + e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_5 + e_6u_6 + e_7u_7),
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial \zeta_0} f_0 = {} & (\frac{\partial u_0}{\partial x_0} + e_1 \frac{\partial u_1}{\partial x_0} + e_2 \frac{\partial u_2}{\partial x_0} - e_3 \frac{\partial u_3}{\partial x_0} + e_4 \frac{\partial u_4}{\partial x_0} + e_5 \frac{\partial u_5}{\partial x_0} + e_6 \frac{\partial u_6}{\partial x_0} + e_7 \frac{\partial u_7}{\partial x_0}) \\
& + (-e_1 \frac{\partial u_0}{\partial x_1} + \frac{\partial u_1}{\partial x_1} - e_3 \frac{\partial u_2}{\partial x_1} + e_2 \frac{\partial u_3}{\partial x_1} - e_5 \frac{\partial u_4}{\partial x_1} + e_4 \frac{\partial u_5}{\partial x_1} + e_7 \frac{\partial u_6}{\partial x_1} - e_6 \frac{\partial u_7}{\partial x_1}) \\
& + (-e_2 \frac{\partial u_0}{\partial x_2} + e_3 \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2} - e_1 \frac{\partial u_3}{\partial x_2} - e_6 \frac{\partial u_4}{\partial x_2} - e_7 \frac{\partial u_5}{\partial x_2} + e_4 \frac{\partial u_6}{\partial x_2} + e_5 \frac{\partial u_7}{\partial x_2}) \\
& + (-e_3 \frac{\partial u_0}{\partial x_3} - e_2 \frac{\partial u_1}{\partial x_3} + e_1 \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_3} - e_7 \frac{\partial u_4}{\partial x_3} + e_6 \frac{\partial u_5}{\partial x_3} - e_5 \frac{\partial u_6}{\partial x_3} + e_4 \frac{\partial u_7}{\partial x_3}) \\
& + (-e_4 \frac{\partial u_0}{\partial x_4} + e_5 \frac{\partial u_1}{\partial x_4} + e_6 \frac{\partial u_2}{\partial x_4} + e_7 \frac{\partial u_3}{\partial x_4} + \frac{\partial u_4}{\partial x_4} - e_1 \frac{\partial u_5}{\partial x_4} - e_2 \frac{\partial u_6}{\partial x_4} - e_3 \frac{\partial u_7}{\partial x_4}) \\
& + (-e_5 \frac{\partial u_0}{\partial x_5} - e_4 \frac{\partial u_1}{\partial x_5} + e_7 \frac{\partial u_2}{\partial x_5} - e_6 \frac{\partial u_3}{\partial x_5} + e_1 \frac{\partial u_4}{\partial x_5} + \frac{\partial u_5}{\partial x_5} + e_3 \frac{\partial u_6}{\partial x_5} - e_2 \frac{\partial u_7}{\partial x_5}) \\
& + (-e_6 \frac{\partial u_0}{\partial x_6} - e_7 \frac{\partial u_1}{\partial x_6} - e_4 \frac{\partial u_2}{\partial x_6} + e_5 \frac{\partial u_3}{\partial x_6} + e_2 \frac{\partial u_4}{\partial x_6} - e_3 \frac{\partial u_5}{\partial x_6} + \frac{\partial u_6}{\partial x_6} + e_1 \frac{\partial u_7}{\partial x_6}) \\
& + (-e_7 \frac{\partial u_0}{\partial x_7} + e_6 \frac{\partial u_1}{\partial x_7} - e_5 \frac{\partial u_2}{\partial x_7} - e_4 \frac{\partial u_3}{\partial x_7} + e_3 \frac{\partial u_4}{\partial x_7} + e_2 \frac{\partial u_5}{\partial x_7} - e_1 \frac{\partial u_6}{\partial x_7} + \frac{\partial u_7}{\partial x_7}),
\end{aligned}
$$

$$
\begin{aligned}
f_0 \frac{\partial}{\partial \zeta_0} = {} & u_0(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3} - e_4 \frac{\partial}{\partial x_4} - e_5 \frac{\partial}{\partial x_5} - e_6 \frac{\partial}{\partial x_6} - e_7 \frac{\partial}{\partial x_7}) \\
& + e_1 u_1(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3} - e_4 \frac{\partial}{\partial x_4} - e_5 \frac{\partial}{\partial x_5} - e_6 \frac{\partial}{\partial x_6} - e_7 \frac{\partial}{\partial x_7}) \\
& + e_2 u_2(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3} - e_4 \frac{\partial}{\partial x_4} - e_5 \frac{\partial}{\partial x_5} - e_6 \frac{\partial}{\partial x_6} - e_7 \frac{\partial}{\partial x_7}) \\
& + e_3 u_3(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3} - e_4 \frac{\partial}{\partial x_4} - e_5 \frac{\partial}{\partial x_5} - e_6 \frac{\partial}{\partial x_6} - e_7 \frac{\partial}{\partial x_7}) \\
& + e_4 u_4(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3} - e_4 \frac{\partial}{\partial x_4} - e_5 \frac{\partial}{\partial x_5} - e_6 \frac{\partial}{\partial x_6} - e_7 \frac{\partial}{\partial x_7}) \\
& + e_5 u_5(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3} - e_4 \frac{\partial}{\partial x_4} - e_5 \frac{\partial}{\partial x_5} - e_6 \frac{\partial}{\partial x_6} - e_7 \frac{\partial}{\partial x_7}) \\
& + e_6 u_6(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3} - e_4 \frac{\partial}{\partial x_4} - e_5 \frac{\partial}{\partial x_5} - e_6 \frac{\partial}{\partial x_6} - e_7 \frac{\partial}{\partial x_7}) \\
& + e_7 u_7)(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3} - e_4 \frac{\partial}{\partial x_4} - e_5 \frac{\partial}{\partial x_5} - e_6 \frac{\partial}{\partial x_6} - e_7 \frac{\partial}{\partial x_7}),
\end{aligned}
$$

$$
\begin{aligned}
f_0 \frac{\partial}{\partial \zeta_0} = {} & (\frac{\partial u_0}{\partial x_0} - e_1 \frac{\partial u_0}{\partial x_1} - e_2 \frac{\partial u_0}{\partial x_2} - e_3 \frac{\partial u_0}{\partial x_3} - e_4 \frac{\partial u_0}{\partial x_4} - e_5 \frac{\partial u_0}{\partial x_5} - e_6 \frac{\partial u_0}{\partial x_6} - e_7 \frac{\partial u_0}{\partial x_7}) \\
& + (e_1 \frac{\partial u_1}{\partial x_0} + \frac{\partial u_1}{\partial x_1} - e_3 \frac{\partial u_1}{\partial x_2} + e_2 \frac{\partial u_1}{\partial x_3} - e_5 \frac{\partial u_1}{\partial x_4} + e_4 \frac{\partial u_1}{\partial x_5} + e_7 \frac{\partial u_1}{\partial x_6} - e_6 \frac{\partial u_1}{\partial x_7}) \\
& + (e_2 \frac{\partial u_2}{\partial x_0} + e_3 \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} - e_1 \frac{\partial u_2}{\partial x_3} - e_6 \frac{\partial u_2}{\partial x_4} - e_7 \frac{\partial u_2}{\partial x_5} + e_4 \frac{\partial u_2}{\partial x_6} + e_5 \frac{\partial u_2}{\partial x_7}) \\
& + (e_3 \frac{\partial u_3}{\partial x_0} - e_2 \frac{\partial u_3}{\partial x_1} + e_1 \frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_3} - e_7 \frac{\partial u_3}{\partial x_4} + e_6 \frac{\partial u_3}{\partial x_5} - e_5 \frac{\partial u_3}{\partial x_6} + e_4 \frac{\partial u_3}{\partial x_7}) \\
& + (e_4 \frac{\partial u_4}{\partial x_0} + e_5 \frac{\partial u_4}{\partial x_1} + e_6 \frac{\partial u_4}{\partial x_2} + e_7 \frac{\partial u_4}{\partial x_3} + \frac{\partial u_4}{\partial x_4} - e_1 \frac{\partial u_4}{\partial x_5} - e_2 \frac{\partial u_4}{\partial x_6} - e_3 \frac{\partial u_4}{\partial x_7}) \\
& + (e_5 \frac{\partial u_5}{\partial x_0} - e_4 \frac{\partial u_5}{\partial x_1} + e_7 \frac{\partial u_5}{\partial x_2} - e_6 \frac{\partial u_5}{\partial x_3} + e_1 \frac{\partial u_5}{\partial x_4} + \frac{\partial u_5}{\partial x_5} + e_3 \frac{\partial u_5}{\partial x_6} - e_2 \frac{\partial u_5}{\partial x_7}) \\
& + (e_6 \frac{\partial u_6}{\partial x_0} - e_7 \frac{\partial u_6}{\partial x_1} - e_4 \frac{\partial u_6}{\partial x_2} + e_5 \frac{\partial u_6}{\partial x_3} + e_2 \frac{\partial u_6}{\partial x_4} - e_3 \frac{\partial u_6}{\partial x_5} + \frac{\partial u_6}{\partial x_6} + e_1 \frac{\partial u_6}{\partial x_7}) \\
& + (e_7 \frac{\partial u_7}{\partial x_0} + e_6 \frac{\partial u_7}{\partial x_1} - e_5 \frac{\partial u_7}{\partial x_2} - e_4 \frac{\partial u_7}{\partial x_3} + e_3 \frac{\partial u_7}{\partial x_4} + e_2 \frac{\partial u_7}{\partial x_5} - e_1 \frac{\partial u_7}{\partial x_6} + \frac{\partial u_7}{\partial x_7}),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \zeta_1}\overline{f_1} = {} & \frac{\partial}{\partial x_8}(u_8 - e_1u_9 - e_2u_{10} - e_3u_{11} - e_4u_{12} - e_5u_{13} - e_6u_{14} - e_7u_{15}) \\
& - e_1\frac{\partial}{\partial x_9}(u_8 - e_1u_9 - e_2u_{10} - e_3u_{11} - e_4u_{12} - e_5u_{13} - e_6u_{14} - e_7u_{15}) \\
& - e_2\frac{\partial}{\partial x_{10}}(u_8 - e_1u_9 - e_2u_{10} - e_3u_{11} - e_4u_{12} - e_5u_{13} - e_6u_{14} - e_7u_{15}) \\
& - e_3\frac{\partial}{\partial x_{11}}(u_8 - e_1u_9 - e_2u_{10} - e_3u_{11} - e_4u_{12} - e_5u_{13} - e_6u_{14} - e_7u_{15}) \\
& - e_4\frac{\partial}{\partial x_{12}}(u_8 - e_1u_9 - e_2u_{10} - e_3u_{11} - e_4u_{12} - e_5u_{13} - e_6u_{14} - e_7u_{15}) \\
& - e_5\frac{\partial}{\partial x_{13}}(u_8 - e_1u_9 - e_2u_{10} - e_3u_{11} - e_4u_{12} - e_5u_{13} - e_6u_{14} - e_7u_{15}) \\
& - e_6\frac{\partial}{\partial x_{14}}(u_8 - e_1u_9 - e_2u_{10} - e_3u_{11} - e_4u_{12} - e_5u_{13} - e_6u_{14} - e_7u_{15}) \\
& - e_7\frac{\partial}{\partial x_{15}}(u_8 - e_1u_9 - e_2u_{10} - e_3u_{11} - e_4u_{12} - e_5u_{13} - e_6u_{14} - e_7u_{15}),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \zeta_1}\overline{f_1} = {} & (\frac{\partial u_8}{\partial x_8} - e_1\frac{\partial u_9}{\partial x_8} - e_2\frac{\partial u_{10}}{\partial x_8} - e_3\frac{\partial u_{11}}{\partial x_8} - e_4\frac{\partial u_{12}}{\partial x_8} - e_5\frac{\partial u_{13}}{\partial x_8} - e_6\frac{\partial u_{14}}{\partial x_8} - e_7\frac{\partial u_{15}}{\partial x_8}) \\
& + (-e_1\frac{\partial u_8}{\partial x_9} - \frac{\partial u_9}{\partial x_9} + e_3\frac{\partial u_{10}}{\partial x_9} - e_2\frac{\partial u_{11}}{\partial x_9} + e_5\frac{\partial u_{12}}{\partial x_9} - e_4\frac{\partial u_{13}}{\partial x_9} - e_7\frac{\partial u_{14}}{\partial x_9} + e_6\frac{\partial u_{15}}{\partial x_9}) \\
& + (-e_2\frac{\partial u_8}{\partial x_{10}} - e_3\frac{\partial u_9}{\partial x_{10}} - \frac{\partial u_{10}}{\partial x_{10}} + e_1\frac{\partial u_{11}}{\partial x_{10}} + e_6\frac{\partial u_{12}}{\partial x_{10}} + e_7\frac{\partial u_{13}}{\partial x_{10}} - e_4\frac{\partial u_{14}}{\partial x_{10}} - e_5\frac{\partial u_{15}}{\partial x_{10}}) \\
& + (-e_3\frac{\partial u_8}{\partial x_{11}} + e_2\frac{\partial u_9}{\partial x_{11}} - e_1\frac{\partial u_{10}}{\partial x_{11}} - \frac{\partial u_{11}}{\partial x_{11}} + e_7\frac{\partial u_{12}}{\partial x_{11}} - e_6\frac{\partial u_{13}}{\partial x_{11}} + e_5\frac{\partial u_{14}}{\partial x_{11}} - e_4\frac{\partial u_{15}}{\partial x_{11}}) \\
& + (-e_4\frac{\partial u_8}{\partial x_{12}} - e_5\frac{\partial u_9}{\partial x_{12}} - e_6\frac{\partial u_{10}}{\partial x_{12}} - e_7\frac{\partial u_{11}}{\partial x_{12}} - \frac{\partial u_{12}}{\partial x_{12}} + e_1\frac{\partial u_{13}}{\partial x_{12}} + e_2\frac{\partial u_{14}}{\partial x_{12}} + e_3\frac{\partial u_{15}}{\partial x_{12}}) \\
& + (-e_5\frac{\partial u_8}{\partial x_{13}} + e_4\frac{\partial u_9}{\partial x_{13}} - e_7\frac{\partial u_{10}}{\partial x_{13}} + e_6\frac{\partial u_{11}}{\partial x_{13}} - e_1\frac{\partial u_{12}}{\partial x_{13}} - \frac{\partial u_{13}}{\partial x_{13}} - e_3\frac{\partial u_{14}}{\partial x_{13}} + e_2\frac{\partial u_{15}}{\partial x_{13}}) \\
& + (-e_6\frac{\partial u_8}{\partial x_{14}} + e_7\frac{\partial u_9}{\partial x_{14}} + e_4\frac{\partial u_{10}}{\partial x_{14}} - e_5\frac{\partial u_{11}}{\partial x_{14}} - e_2\frac{\partial u_{12}}{\partial x_{14}} + e_3\frac{\partial u_{13}}{\partial x_{14}} - \frac{\partial u_{14}}{\partial x_{14}} - e_1\frac{\partial u_{15}}{\partial x_{14}}) \\
& + (-e_7\frac{\partial u_8}{\partial x_{15}} - e_6\frac{\partial u_9}{\partial x_{15}} + e_5\frac{\partial u_{10}}{\partial x_{15}} + e_4\frac{\partial u_{11}}{\partial x_{15}} - e_3\frac{\partial u_{12}}{\partial x_{15}} - e_2\frac{\partial u_{13}}{\partial x_{15}} + e_1\frac{\partial u_{14}}{\partial x_{15}} - \frac{\partial u_{15}}{\partial x_{15}}),
\end{aligned}
$$

$$
\begin{aligned}
f_1\frac{\partial}{\partial \overline{\zeta_1}} = {} & u_8(\frac{\partial}{\partial x_8} + e_1\frac{\partial}{\partial x_9} + e_2\frac{\partial}{\partial x_{10}} + e_3\frac{\partial}{\partial x_{11}} + e_4\frac{\partial}{\partial x_{12}} + e_5\frac{\partial}{\partial x_{13}} + e_6\frac{\partial}{\partial x_{14}} + e_7\frac{\partial}{\partial x_{15}}) \\
& + e_1u_9(\frac{\partial}{\partial x_8} + e_1\frac{\partial}{\partial x_9} + e_2\frac{\partial}{\partial x_{10}} + e_3\frac{\partial}{\partial x_{11}} + e_4\frac{\partial}{\partial x_{12}} + e_5\frac{\partial}{\partial x_{13}} + e_6\frac{\partial}{\partial x_{14}} + e_7\frac{\partial}{\partial x_{15}}) \\
& + e_2u_{10}(\frac{\partial}{\partial x_8} + e_1\frac{\partial}{\partial x_9} + e_2\frac{\partial}{\partial x_{10}} + e_3\frac{\partial}{\partial x_{11}} + e_4\frac{\partial}{\partial x_{12}} + e_5\frac{\partial}{\partial x_{13}} + e_6\frac{\partial}{\partial x_{14}} + e_7\frac{\partial}{\partial x_{15}}) \\
& + e_3u_{11}(\frac{\partial}{\partial x_8} + e_1\frac{\partial}{\partial x_9} + e_2\frac{\partial}{\partial x_{10}} + e_3\frac{\partial}{\partial x_{11}} + e_4\frac{\partial}{\partial x_{12}} + e_5\frac{\partial}{\partial x_{13}} + e_6\frac{\partial}{\partial x_{14}} + e_7\frac{\partial}{\partial x_{15}}) \\
& + e_4u_{12}(\frac{\partial}{\partial x_8} + e_1\frac{\partial}{\partial x_9} + e_2\frac{\partial}{\partial x_{10}} + e_3\frac{\partial}{\partial x_{11}} + e_4\frac{\partial}{\partial x_{12}} + e_5\frac{\partial}{\partial x_{13}} + e_6\frac{\partial}{\partial x_{14}} + e_7\frac{\partial}{\partial x_{15}}) \\
& + e_5u_{13}(\frac{\partial}{\partial x_8} + e_1\frac{\partial}{\partial x_9} + e_2\frac{\partial}{\partial x_{10}} + e_3\frac{\partial}{\partial x_{11}} + e_4\frac{\partial}{\partial x_{12}} + e_5\frac{\partial}{\partial x_{13}} + e_6\frac{\partial}{\partial x_{14}} + e_7\frac{\partial}{\partial x_{15}}) \\
& + e_6u_{14}(\frac{\partial}{\partial x_8} + e_1\frac{\partial}{\partial x_9} + e_2\frac{\partial}{\partial x_{10}} + e_3\frac{\partial}{\partial x_{11}} + e_4\frac{\partial}{\partial x_{12}} + e_5\frac{\partial}{\partial x_{13}} + e_6\frac{\partial}{\partial x_{14}} + e_7\frac{\partial}{\partial x_{15}}) \\
& + e_7u_{15})(\frac{\partial}{\partial x_8} + e_1\frac{\partial}{\partial x_9} + e_2\frac{\partial}{\partial x_{10}} + e_3\frac{\partial}{\partial x_{11}} + e_4\frac{\partial}{\partial x_{12}} + e_5\frac{\partial}{\partial x_{13}} + e_6\frac{\partial}{\partial x_{14}} + e_7\frac{\partial}{\partial x_{15}}),
\end{aligned}
$$

and

$$
\begin{aligned}
f_1 \frac{\partial}{\partial \overline{\zeta_1}} = &\left(\frac{\partial u_8}{\partial x_8} + e_1 \frac{\partial u_8}{\partial x_9} + e_2 \frac{\partial u_8}{\partial x_{10}} + e_3 \frac{\partial u_8}{\partial x_{11}} + e_4 \frac{\partial u_8}{\partial x_{12}} + e_5 \frac{\partial u_8}{\partial x_{13}} + e_6 \frac{\partial u_8}{\partial x_{14}} + e_7 \frac{\partial u_8}{\partial x_{15}}\right) \\
&+ e_1\left(\frac{\partial u_9}{\partial x_8} + e_1 \frac{\partial u_9}{\partial x_9} + e_2 \frac{\partial u_9}{\partial x_{10}} + e_3 \frac{\partial u_9}{\partial x_{11}} + e_4 \frac{\partial u_9}{\partial x_{12}} + e_5 \frac{\partial u_9}{\partial x_{13}} + e_6 \frac{\partial u_9}{\partial x_{14}} + e_7 \frac{\partial u_9}{\partial x_{15}}\right) \\
&+ e_2\left(\frac{\partial u_{10}}{\partial x_8} + e_1 \frac{\partial u_{10}}{\partial x_9} + e_2 \frac{\partial u_{10}}{\partial x_{10}} + e_3 \frac{\partial u_{10}}{\partial x_{11}} + e_4 \frac{\partial u_{10}}{\partial x_{12}} + e_5 \frac{\partial u_{10}}{\partial x_{13}} + e_6 \frac{\partial u_{10}}{\partial x_{14}} + e_7 \frac{\partial u_{10}}{\partial x_{15}}\right) \\
&+ e_3\left(\frac{\partial u_{11}}{\partial x_8} + e_1 \frac{\partial u_{11}}{\partial x_9} + e_2 \frac{\partial u_{11}}{\partial x_{10}} + e_3 \frac{\partial u_{11}}{\partial x_{11}} + e_4 \frac{\partial u_{11}}{\partial x_{12}} + e_5 \frac{\partial u_{11}}{\partial x_{13}} + e_6 \frac{\partial u_{11}}{\partial x_{14}} + e_7 \frac{\partial u_{11}}{\partial x_{15}}\right) \\
&+ e_4\left(\frac{\partial u_{12}}{\partial x_8} + e_1 \frac{\partial u_{12}}{\partial x_9} + e_2 \frac{\partial u_{12}}{\partial x_{10}} + e_3 \frac{\partial u_{12}}{\partial x_{11}} + e_4 \frac{\partial u_{12}}{\partial x_{12}} + e_5 \frac{\partial u_{12}}{\partial x_{13}} + e_6 \frac{\partial u_{12}}{\partial x_{14}} + e_7 \frac{\partial u_{12}}{\partial x_{15}}\right) \\
&+ e_5\left(\frac{\partial u_{13}}{\partial x_8} + e_1 \frac{\partial u_{13}}{\partial x_9} + e_2 \frac{\partial u_{13}}{\partial x_{10}} + e_3 \frac{\partial u_{13}}{\partial x_{11}} + e_4 \frac{\partial u_{13}}{\partial x_{12}} + e_5 \frac{\partial u_{13}}{\partial x_{13}} + e_6 \frac{\partial u_{13}}{\partial x_{14}} + e_7 \frac{\partial u_{13}}{\partial x_{15}}\right) \\
&+ e_6\left(\frac{\partial u_{14}}{\partial x_8} + e_1 \frac{\partial u_{14}}{\partial x_9} + e_2 \frac{\partial u_{14}}{\partial x_{10}} + e_3 \frac{\partial u_{14}}{\partial x_{11}} + e_4 \frac{\partial u_{14}}{\partial x_{12}} + e_5 \frac{\partial u_{14}}{\partial x_{13}} + e_6 \frac{\partial u_{14}}{\partial x_{14}} + e_7 \frac{\partial u_{14}}{\partial x_{15}}\right) \\
&+ e_7\left(\frac{\partial u_{15}}{\partial x_8} + e_1 \frac{\partial u_{15}}{\partial x_9} + e_2 \frac{\partial u_{15}}{\partial x_{10}} + e_3 \frac{\partial u_{15}}{\partial x_{11}} + e_4 \frac{\partial u_{15}}{\partial x_{12}} + e_5 \frac{\partial u_{15}}{\partial x_{13}} + e_6 \frac{\partial u_{15}}{\partial x_{14}} + e_7 \frac{\partial u_{15}}{\partial x_{15}}\right).
\end{aligned}
$$

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