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Blow-Up Time of Solutions for a Parabolic Equation with Exponential Nonlinearity

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Abstract: This paper studies a parabolic equation with exponential nonlinearity, which has several applications, for example the self-trapped beams in plasma. Based on a modified concavity method we prove the blow-up of the solution for initial data with high initial energy. We also proposed the solution's lower and upper bound of the blow-up time for the equation. Our results complement the existing results.

Keywords: parabolic equation; exponential nonlinearity; blow-up

MSC: 35L20; 35L70; 35B30

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1. Introduction

This paper is concerned with blow-up of solutions for the following parabolic equation with exponential nonlinear source

$$\begin{cases} u_t = \Delta u - u + \lambda f(u) \ in \ (0,T) \times \ \mathbb{R}^2 \\ u(x,0) = u_0(x) in \mathbb{R}^2, \end{cases} \tag{1}$$

where $\lambda > 0$,

$$f(u) := 2\alpha_0 u e^{\alpha_0 u^2}, \text{ for some } \alpha_0 > 0, \tag{2}$$

with initial data $u_0 \in H^1(\mathbb{R}^2)$. Equation (1) has several applications, for example, the self-trapped beams in plasma [1]. Moreover, the two-dimensional case is interesting because of its relation to the critical Moser–Trudinger inequalities [2,3].

Equation (1) is well known with power-type nonlinearity as $f(u) = |u|^{p-1}u$, which has been extensively studied [4]. The model is used to study the competition between the dissipative of diffusion and the influence of an explosive source term. The first result with singular initial data is due to Weissler [5,6]. Messaoudi [7] and Liu and Wang [8] both studied the Cauchy problem with vanishing and positive initial energy blow-up for some special parabolic equations in finite time, respectively. Furthermore, for Equation (1) with power-type nonlinearity and a memory term, the finite-time blow-up result for the solution has been proved with positive initial energy in [9]. Tian [10] given out the bound of blow-up time of the viscoelastic parabolic equation. Furthermore, the blow-up bounds of Equation (1) with different nonlinearities except exponential nonlinearity were studied in [11,12]. Moreover, it is noted that analytic methods were numerically used to study various one-dimensional parabolic Equations [13–16].

In the past decades, more and more attention has been devoted to the blow-up study of wave equations with arbitrarily initial energy [17–19]. Recently, the blow-up bounds of wave equations with various nonlinearities have been studied [20,21]. Nevertheless, the proof cannot directly apply to the parabolic equations.

In this paper, we focus on Equation (1) with the exponential nonlinearity (2). If λ satisfies the following condition

$$0 < \lambda < \frac{1}{2\alpha_0}, \tag{3}$$

then the existence of ground solutions for the stationary problem associated with (1) has been proved in [22]. For the case $\lambda \geq \frac{1}{2\alpha_0}$ the corresponding stationary problem has no non-trivial $H^1(\mathbb{R}^2)$- solution. As in [23], we define the maximal existence time T_* of the solution $u(x,t)$ as

$$T_* := \sup\Big\{T > 0 |\ \text{the problem (1) admits a solution}\ u \in C\Big([0,T]; H^1(\mathbb{R}^2)\Big)\Big\} \in (0, +\infty].$$

In order to introduce some existing results for the problem (1), we now denote some notations: We use $\|\cdot\|_q$ to denote the norm in $L^q(\Omega)$. For simplicity, we always use $\|\cdot\|$ to denote $\|\cdot\|_2$. Furthermore, let $H(\Omega)$ be the Sobolev space with the norm as $(\|\nabla(\cdot)\|^2 + \|\cdot\|^2)^{\frac{1}{2}}$. We next define two auxiliary functions.

$$J(v) := \frac{1}{2}\|v\|_{H^1}^2 - \lambda \int_{\mathbb{R}^2} F(v)dx, \tag{4}$$

where

$$F(v) := \int_0^v f(\mu)d\mu = e^{\alpha_0 v^2} - 1,$$

and

$$I(v) := \|v\|_{H^1}^2 - \lambda \int_{\mathbb{R}^2} vf(v)dx. \tag{5}$$

The potential well and its corresponding set are defined, respectively, by

$$W := \{u \in H^1(\mathbb{R}^2) | I(u) > 0, J(u) < d\} \cup \{0\}, \tag{6}$$

$$V := \{u \in H^1\mathbb{R}^2 | I(u) < 0, J(u) < d\}, \tag{7}$$

where the depth d of the potential well is characterized by

$$d := \inf_{u \in H^1(\mathbb{R}^2)\setminus\{0\}} \sup_{\omega \geq 0} J(\omega u) = \inf_{u \in \mathcal{N}} J(u), \tag{8}$$

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^2) | I(u) = 0, \|\nabla u\| \neq 0\}. \tag{9}$$

Concerning local existence and uniqueness for Equation (1) in [23–25], for any $u_0 \in H^1(\mathbb{R}^2)$ the Cauchy problem (1) has a unique local in time solution $u \in C([0,T]; H^1(\mathbb{R}^2))$ for some finite time $T > 0$.

In [23,24], the main results of global existence and non-existence of solutions for Equation (1) can be summarized as follows: when $J(u_0) < d$,

(i) if $u_0 \in W$ then the maximal solutions to (1) with λ as in (3) exist globally;
(ii) if $u_0 \in V$ then the maximal solutions to (1) with λ as in (3) blow up in finite time.

As we know, the existing blow-up result did not consider the case of arbitrarily high initial energy for Equation (1). This paper is devoted to studying the blow-up result for the parabolic Equation (1) with exponential nonlinearity and high initial energy.

Under $u_0 \in H^1(\mathbb{R}^2)$, the local existence has been proved in [23], then we are in a position to state our main blow-up result for Equation (1).

Theorem 1. *Let $u \in C([0,T); H^1(\mathbb{R}^2))$ be the maximal solution to (1) with λ satisfying*

$$0 < \lambda < \frac{1}{4\alpha_0}, \tag{10}$$

and $u_0 \in H^1(\mathbb{R}^2)$. If

$$J(u_0) < \frac{\theta - 2}{2\theta}\xi_1 \|u_0\|^2, \tag{11}$$

where $2 < \theta < 2(1+\varepsilon)$, $\varepsilon = \frac{1}{4\lambda\alpha_0} - 1$, and $\xi_1 > 0$ is the largest eigenvalue of $-\Delta$ in $\mathbb{R}^2$ with homogeneous Dirichlet boundary condition. Then the solution $u(x,t)$ blows up at a sufficiently large time T, where T has an upper bound $\bar{T}$ as (36).

Moreover, if u_0 also satisfies that

$$\|\nabla u_0\|^2 < \frac{4\pi}{\alpha_0(1+\epsilon)}, \tag{12}$$

$$\|u_0\|^2 < M \tag{13}$$

for some $\epsilon > 0$ and $M > 0$, then the blow-up time has a lower bound $\underline{T}$ as (41).

Remark 1. *The energy $J(u)$ may be arbitrarily high. We next show this by a contradiction. Suppose that there exists some $M > 0$ such that the energy $J(u) < M$ for any u. Here we can suppose that $u \in L_\infty(\mathbb{R}^2)$. If $u \notin L_\infty(\mathbb{R}^2)$, then we assume that there exists a $\bar{x} \in \mathbb{R}^2$ such that $u \to \infty$ as $x \to \bar{x}$. We set*

$$\tilde{u} = \begin{cases} u, x \in \mathbb{R}^2 \ \bar{x}(\epsilon), \\ 0, x \in \bar{x}(\epsilon), \end{cases}$$

where $\bar{x}(\epsilon) = \{x \in \mathbb{R}^2 : \|x - \bar{x}\| < \epsilon\}$, ϵ is some small positive number.

Then, for every $u \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, it holds in [26]

$$\int_{\mathbb{R}^2} (e^{\alpha|u(x)|^2} - 1)dx \leq \frac{1}{\sqrt{\log 2}}(\|u\| + \|u\|_\infty). \tag{14}$$

We now take $u_0 = ru$ for some $r > 0$. Then we see that

$$\begin{aligned} J(ru) &= \frac{1}{2}\Big(r^2\|u\|^2 + r^2\|\nabla u\|^2\Big) - \lambda \int_{\mathbb{R}^2} \Big(e^{\alpha r^2 u^2} - 1\Big)dx \\ &\geq \frac{1}{2}\Big(r^2\|u\|^2 + r^2\|\nabla u\|^2\Big) - \frac{1}{\sqrt{\log 2}} r(\|u\| + \|u\|_\infty)dx. \end{aligned}$$

If $r = 1$, then $\frac{1}{2}(r^2\|u\|^2 + r^2\|\nabla u\|^2) - \frac{1}{\sqrt{\log 2}} r(\|u\| + \|u\|_\infty)dx < M$. Obviously, as $r \to +\infty$, $\frac{1}{2}(r^2\|u\|^2 + r^2\|\nabla u\|^2) - \frac{1}{\sqrt{\log 2}} r(\|u\| + \|u\|_\infty)dx \to +\infty$, which implies that for arbitrarily high initial energy, there exists u_0 satisfying (11).

2. Proof of Theorem 1

We first state the following equalities, which have been proved in [23]

$$\frac{d}{dt}J(u(t)) = -\|\partial_t u(t)\|^2, \tag{15}$$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = -I(u(t)). \tag{16}$$

In our proof we need the following auxiliary growth functions

$$\tilde{f}(u) := 2\alpha_0 u(e^{\alpha_0 u^2} - 1), \text{ and } \tilde{F}(u) := e^{\alpha_0 u^2} - 1 - \alpha_0 u^2. \tag{17}$$

It is obvious that

$$f(u) = \tilde{f} + 2\alpha_0 u, \text{ and } F(u) = \tilde{F}(u) + \alpha_0 u^2.$$

By a direct computation, we see that for any $\theta > 2$, it is satisfied that

$$\theta\tilde{F}(s) \leq s\tilde{f}(s), \text{ for any } s \in \mathbb{R}. \tag{18}$$

In order to prove the blow-up result, the following lemma is necessary, which has been proved in [27].

Lemma 1. *If $\Phi(t)$ is a nonincreasing function on $[0, +\infty]$, and satisfies that*

$$\Phi'(t)^2 \geq a + b\Phi(t)^{2+\frac{1}{\kappa}} \tag{19}$$

for $t \geq 0$, where $a > 0$ and $b > 0$, then there exists a finite time $T^* > 0$ such that

$$lim_{t\to T^{*-}}\Phi(t) = 0,$$

where

$$T^* \leq 2^{\frac{3\kappa+1}{2\kappa}} \frac{\kappa(\frac{a}{b})^{2+1/\kappa}}{\sqrt{a}}(1 - (1 + (\frac{a}{b})^{2+1/\kappa}\Phi(0)))^{-1/(2\kappa)}.$$

2.1. Proof of the Upper Bound of Blow-Up Time in Theorem 1

We first prove that $I(u_0) < 0$. By (4) and (5),

$$\begin{aligned}
I(u_0) &= \theta J(u_0) - \frac{\theta-2}{2}(\|u_0\|^2 + \|\nabla u_0\|^2) - \lambda\int_{\mathbb{R}^2} u_0 f(u_0)dx + \lambda\theta\int_{\mathbb{R}}^2 F(u_0)dx \\
&= \theta\left(J(u_0) - \frac{\theta-2}{2\theta}\xi_1\|u_0\|^2\right) - \left(\frac{\theta-2}{2} - \lambda(\theta\alpha_0 - 2\alpha_0)\right)\|u_0\|^2 \\
&\quad - \frac{\theta-2}{2}\left(\|\nabla u_0\|^2 - \xi_1\|u_0\|^2\right) + \lambda\left(\int_{\mathbb{R}^2}\theta\tilde{F}(u_0)dx - \int_{\mathbb{R}^2}u_0\tilde{f}(u_0)dx\right).
\end{aligned} \tag{20}$$

By $\theta > 2$ and (10), we can easily see that

$$\left(\frac{\theta-2}{2} - \lambda(\theta\alpha_0 - 2\alpha_0)\right) > 0. \tag{21}$$

By (11), (21), (18) and the definition of ξ_1, we have

$$I(u_0) < 0. \tag{22}$$

Next, by a contradiction argument, we prove that

$$I(u(t)) < 0$$

for all $t \in [0, T)$.

Suppose that there exists a time t_1 such that

$$t_1 := min\{t \in (0, T) : I(u) = 0\} > 0.$$

Following the local existence results in [23], we see that $u(t, x)$ is continuous as a function of t. Then we see that $I(u(t)) < 0$ when $t \in [0, t_1)$ and $I(u(t_1)) = 0$. By (16) we have that $\|u(t)\|^2$ is strictly increasing in t for $t \in [0, t_1)$, thus,

$$0 \leq J(u_0) < \frac{\theta-2}{2\theta}\xi_1\|u_0\|^2 < \frac{\theta-2}{2\theta}\xi_1\|u(t_1)\|^2. \tag{23}$$

On the other hand, it follows from (4), (15) and the fact that $I(u(t_1) = 0$ that

$$\begin{aligned} J(u_0) \geq J(u(t_1)) &= \frac{1}{2}(\|u(t_1)\|^2 + \|\nabla u(t_1)\|^2) - \lambda \int_{\mathbb{R}^2} F(u(t_1))dx - \frac{1}{\theta} I(u(t_1)) \\ &= \left(\frac{\theta - 2}{2\theta} - \lambda(\alpha_0 \theta - 2\alpha_0) \right) \|u(t_1)\|^2 + \frac{\theta - 2}{2\theta} \|\nabla u(t_1)\|^2 \\ &+ \frac{\lambda}{\theta} \left(\int_{\mathbb{R}^2} u(t_1) \tilde{f}(u(t_1))dx - \theta \int_{\mathbb{R}^2} \tilde{F}(u(t_1))dx \right) \\ &\geq \frac{\theta - 2}{2\theta} \|\nabla u(t_1)\|^2 \\ &\geq \frac{\theta - 2}{2\theta} \xi_1 \|u(t_1)\|^2, \end{aligned} \tag{24}$$

which contradicts (23). Thus, we have proved that $I(u(t)) < 0$ for all $t \in [0, T)$.

Furthermore, by (15) we see that the following is always valid on $[0, T)$

$$J(u(t)) < \frac{\theta - 2}{2\theta} \xi_1 \|u_0\|^2.$$

Secondly, we prove that the solution of Equation (1) blows up in a finite time. We now suppose that T^* is sufficiently large. Then, we define the following auxiliary function: for $t \in [0, T^*)$

$$G(t) = \int_0^t \|u(\tau)\|^2 d\tau + (T^* - t)\|u_0\|^2 + a(t + \sigma)^2, \tag{25}$$

with $T^* \in (0, T)$, $a > 0$ and $\sigma > 0$.

We can obtain

$$\begin{aligned} G'(t) &= \|u(t)\|^2 - \|u_0\|^2 + 2a(t + \sigma) \\ &= \int_0^t \frac{d}{d\tau} \|u(\tau)\|^2 d\tau + 2a(t + \sigma), \end{aligned} \tag{26}$$

and

$$\begin{aligned} \frac{1}{2} G''(t) &= \frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + a \\ &= -I(u(t)) + a \\ &\geq -\left(\|\nabla u(t)\|^2 + (1 - 2\alpha_0 \lambda) \|u(t)\|^2 \right) + \lambda \theta \int_{\mathbb{R}^2} \tilde{F}(u(t))dx + a \\ &\geq -\theta J(u(t) + \frac{\theta - 2}{2} \left(\|\nabla u\|^2 + (1 - 2\alpha_0 \lambda) \|u(t)\|^2 \right) + a \\ &\geq -\theta J(u(t)) + (\theta - 2)(1 - 2\alpha_0 \lambda) \|u(t)\|^2 + a \\ &= -\theta J(u_0) + \int_0^t \|u_\tau(\tau)\| d\tau + (\theta - 2)(1 - 2\alpha_0 \lambda) \|u(t)\|^2 + a, \end{aligned} \tag{27}$$

where the penultimate inequality follows from Poincaré inequality [28]. Obviously, we can choose a sufficient large a such that $a - \theta J(u_0) > 0$, which means that $G''(t) > 0$ for every $t \in (0, T)$. Since $G'(0) = 2a\sigma > 0$, then $G'(t) > 0$ for every $t \in [0, T^*)$. Thus, we see that $G(t)$ is strictly increasing on $[0, T^*)$. As $G(0) = T^* \|u_0\|^2 + a\sigma^2 > 0$, we have that $G(t) > 0$ for any $t \in [0, T^*)$.

Now we denote

$$
\begin{aligned}
A &:= \int_0^t \|u(\tau)\|^2 d\tau + a(t+\sigma)^2, &(28)\\
B &:= \frac{1}{2}\int_0^t \frac{d}{d\tau}\|u(\tau)\|^2 d\tau + a(t+\sigma), &(29)\\
C &:= \int_0^t \|u_\tau(\tau)\|^2 + a. &(30)
\end{aligned}
$$

Additionally, we see that for any $s \in \mathbb{R}$

$$
\begin{aligned}
As^2 - 2Bs + C &= \int_0^t \|(su(\tau) - u_\tau(\tau))\|^2 d\tau + a((t+\sigma)s+1)^2\\
&\geq 0,
\end{aligned}
$$

which implies that $AC - B^2 \leq 0$.

Furthermore, we have

$$
\begin{aligned}
G(t) &\geq A,\\
G'(t) &= 2B,\\
G''(t) &\geq 2\theta C - (2\theta-1)a - 2\theta J(u_0).
\end{aligned}
$$

Then,

$$
\begin{aligned}
G(t)G''(t) - \frac{\theta}{2}(G'(t))^2 &\geq 2\theta(AC - B^2) - ((2\theta-1)a + 2\theta J(u_0))G(t)\\
&\geq -((2\theta-1)a + 2\theta J(u_0))G(t).
\end{aligned}
$$

Define another auxiliary function

$$
\Phi(t) = (G(t))^{-\frac{\theta}{2}+1}. \tag{31}
$$

By direct computation, we have

$$
\begin{aligned}
\Phi'(t) &= \left(-\frac{\theta}{2}+1\right)(G(t))^{-\frac{\theta}{2}}G'(t), &(32)\\
\Phi''(t) &= \left(-\frac{\theta}{2}+1\right)(G(t))^{-\frac{\theta}{2}-1}\left(G(t)G''(t) - \frac{\theta}{2}(G'(t))^2\right) &(33)\\
&\leq ((2\theta-1)a + 2\theta J(u_0))\left(\frac{\theta}{2}-1\right)(\Phi(t))^{\frac{\theta}{\theta-2}}. &(34)
\end{aligned}
$$

By the facts, $G(t) > 0$ and $G'(t) > 0$, we see that $\Phi'(t) < 0$. Thus, by multiplying (34) by $\Phi'(t)$ and integrating it from 0 to t, we obtain that

$$
\Phi'(t) \geq C_0 + C_1\Phi(t)^{\frac{2\theta-2}{\theta-2}}, \tag{35}
$$

where

$$
\begin{aligned}
C_0 &= \Phi^2(0) - ((2\theta-1)a + 2\theta J(u_0))(2\theta-2)\Phi(0)^{\frac{2\theta-2}{\theta-2}},\\
C_1 &= ((2\theta-1)a + 2\theta J(u_0))(2\theta-2).
\end{aligned}
$$

If we want $C_0 > 0$, by direct calculation, it is sufficient that

$$
T^*\|u_0\|^2 + a\sigma^2 - ((2\theta-1)a + 2\theta J(u_0))(2\theta-2) > 0.
$$

There exists a $\sigma > 0$ sufficiently large such that the above inequality is valid.

Now from Lemma 1 it is following that there exists a finite time $T > 0$ such that $\Phi(t) \to 0$ as $t \to T^-$, which means that $G(t) \to +\infty$ as $t \to T^-$

By Lemma 1 we estimate the upper bound $\bar{T}$ of the blow-up time T with $T < \bar{T}$, where

$$\bar{T} = 2^{\frac{3\theta-4}{2(\theta-2)}} \frac{\frac{\theta-2}{2}\left(\frac{C_0}{C_1}\right)^{\frac{2\theta-2}{\theta-2}}}{C_0^{1/2}} \left(1 - \left(1 + \frac{C_0}{C_1}\right)^{\frac{2\theta-2}{\theta-2}} \Phi(0)\right)^{-\frac{1}{\theta-2}}. \tag{36}$$

2.2. *Proof of the Lower Bound of blow-up Time in Theorem 1*

We define the function

$$\phi(x) = \frac{1}{2} \int_{\mathbb{R}^2} u^2(x,t)dx. \tag{37}$$

By (16) we see that

$$\begin{aligned} \phi'(x) &= -I(t) \\ &\leq \lambda \int_{\mathbb{R}^2} uf(u)dx \\ &= \lambda\left(2\alpha_0\|u\|^2 + \int_{\mathbb{R}^2} 2\alpha_0 u^2(e^{\alpha_0 u^2} - 1)dx\right). \end{aligned} \tag{38}$$

Equation (1) can be written in the equivalent integral formulation

$$\Gamma : u(t) = e^{t\Delta}u_0 - \lambda \int_0^t e^{(t-s)\Delta} f(u(s))ds.$$

Define the following set

$$X = \left\{ u \in L((0,T), H^1(\mathbb{R}^2)) : \sup_{t\in[0,T)} \|\nabla u(t)\|^2 \leq \frac{4\pi}{\alpha_0(1+\frac{\epsilon}{2})}, \sup_{t\in[0,T)} \|u(t)\|^2 \leq \beta M \right\},$$

where β is some positive constant.

Following the proof in [23] we see that Γ maps X into itself and have the next estimation

$$\|u(t)\| \quad \leq \quad \sqrt{M} + C_\epsilon t\sqrt{(}M) + C_r \int_0^t t^{3/2-1/r},$$

where $1 < r < 2$. As $T < \bar{T}$, and we take $r = 3/2$, then we obtain

$$\|u(t)\| \quad \leq \quad C_{\epsilon,M,\bar{T}}. \tag{39}$$

By the scale-invariant Trudinger–Moser inequality, we can obtain that for any $v \in H^1(\mathbb{R}^2)$,

$$\begin{aligned} \int_{\mathbb{R}^2} u^2(x,t)(e^{2\alpha_0 u^2(x,t)} - 1)dx &\leq \left(\int_{\mathbb{R}^2} |u(x,t)|^{2q_1}\right)^{1/q_1} \left(\int_{\mathbb{R}^2} (e^{\alpha_0 q_2 u^2(x,t)} -)dx\right)^{1/q_2} \\ &\leq C\|u(t)\|^2 \left(\int_{\mathbb{R}^2} (e^{\alpha_0 q_2 u^2(x,t)} -)dx\right)^{1/q_2} \\ &\leq C\left(\frac{4\pi}{\alpha_0(1+\epsilon)}\right)^{-1/(1+\epsilon)} \|u\|^{2(2+\epsilon)}, \end{aligned} \tag{40}$$

where we use Holder inequality with $q_1, q_2 > 1$ satisfying $\frac{1}{q_1} + \frac{1}{q_2} = 1$ and $q_2 = 1 + \epsilon$, and Gagliardo–Nirenberg inequality as following

$$\|v\|_{L^q}^q \leq C\|\nabla v\|^{q-2}\|v\|^2$$

for any $v \in H^1(\mathbb{R}^2)$ with $q \geq 2$.

Combining (38) and (40) we obtain that

$$\begin{aligned}\phi'(t) &= C\left(\frac{4\pi}{\alpha_0(1+\epsilon)}\right)^{-1/(1+\epsilon)}(C_{\epsilon,M,\bar{T}} + \phi^{2+\epsilon}(t)) \\ &\leq C\left(\frac{4\pi}{\alpha_0(1+\epsilon)}\right)^{-1/(1+\epsilon)}(C_{\epsilon,M,\bar{T}} + \phi(t))^{2+\epsilon}\end{aligned}$$

for $t \in [0, T)$.

Since $I(u(t)) < 0$ then $\phi(t) > 0$ for any $t \in [0, T)$. Thus, we see that there exists a lower bound $underlineT$ of blow-up time for Equation (1)

$$\begin{aligned}t \geq \underline{T} &= \left(\frac{4\pi}{\alpha_0(1+\epsilon)}\right)^{1/(1+\epsilon)}\frac{\|u_0\|^{-2(1+\epsilon)}}{(1+\epsilon)C} \\ &= \left(\frac{4\pi}{\alpha_0(1+\epsilon)}\right)^{1/(1+\epsilon)}\frac{M^{-(1+\epsilon)}}{(1+\epsilon)C}.\end{aligned} \tag{41}$$

Thus, the proof of Theorem 1 is completed.

3. Conclusions

This paper studies the bound of blow-up time for the parabolic equation with exponential nonlinearity. This paper proved the blow-up time bound for the parabolic equation with exponential nonlinearity based on the modified concavity method. Up to our knowledge [23,24], this result is new for the exponential parabolic equations. Furthermore, we note that some studies use analytic methods to study the parabolic equations with various nonlinearities in $\mathbb{R}^1$ [13–16], which provides the main impetus for our future study.

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References

1. Lam, J.F.; Lippman, B.; Trappert, F. Sel trapped lased beams in plasma. *Phys. Fluid* **1997**, *20*, 1176–1179. [CrossRef]
2. Adachi, S.; Tanaka, K. Trudinger type inequalities in $\mathbb{R}^N$ and their best exponent. *Proc. Am. Math. Soc.* **1999**, *128*, 2051–2057. [CrossRef]
3. Ruf, B. A sharp Moser-Trudinger type inequality for unbounded domains in $\mathbb{R}^2$. *J. Funct. Anal.* **2004**, *219*, 304–367. [CrossRef]
4. Li, T.T.; Mimura, M.; Nishiura, Y.; Ye, Q.X. Reaction-diffusion equations and their applications and computations aspects. In Proceedings of the China-Japan Symposium, Shanghai, China, 30 October–4 November 1994; World Scientific: Singapore, 1997.
5. Weissler, F.B. Local existence and nonexistence for a semilinear parabolic equation in L^p. *Indiana Univ. Math. J.* **1980**, *29*, 79–102. [CrossRef]
6. Weissler, F.B. Existence and nonexistence of global solutions for a semilinear heat equation. *Isr. J. Math.* **1981**, *38*, 29–40. [CrossRef]
7. Messaoudi, S.A. A note on blow up of solutions of a quasilinear heat equation with vanishing initial energy. *J. Math. Anal. Appl.* **2002**, *273*, 243–247. [CrossRef]
8. Liu, G.W.; Chen, H. Global and blow-up of solutions for a quasilinear parabolic system with viscoelastic and source term. *Math. Methods Appl. Sci.* **2014**, *37*, 148–156. [CrossRef]
9. Liu, W.J.; Wang, M.X. Blow-up of the solution for a p-Laplacian equation with positive initial energy. *Acta. Appl. Math.* **2008**, *103*, 141–146. [CrossRef]
10. Tian, S.Y. Bounds for blow-up time in a semilinear parabolic problem with viscoelastic term. *Comp. Math. Appl.* **2017**, *74*, 736–743. [CrossRef]

11. Zhou, J. Blow-up and lifespan of solutions to a nonlocal parabolic equation at arbitrary initial energy level. *Appl. Math. Lett.* **2018**, *78*, 118–125. [CrossRef]
12. Han, Y.; Gao, W.; Sun, Z.; Li, H. Upper and lower bounds of blow-up time to a parabolic type Kirchhoff equation with arbitrary initial energy. *Comput. Math. Appl.* **2018**, *76*, 2477–2483. [CrossRef]
13. Ahmadian, S.; Darvishi, M.T. A new fractional Biswas-Milovic model with its periodic soliton solutions. *Optik* **2016**, *127*, 7694–7703. [CrossRef]
14. Zaidan, L.I.; Darvishi, M.T. Numerical simulation for fractional nonliear (1+1)-dimensional Biswas-Milovic eqution. *Optik* **2017**, *136*, 390–402. [CrossRef]
15. Palencia, J.L.D.; Rahman, S.U.; Redondo, A.N. Heterogeneous diffusion and nonlinear advection in a one-dimension FIsher-KPP problem. *Entropy* **2022**, *24*, 915. [CrossRef]
16. Ahamd, S.; Ullah, A.; Akgül, A.; Jarad, F. A hybrid analytical techniquel for solving nonlinear fraction order PDEs of power law kernel: Application to KdV and Forberg-Witham equations. *AIMS Math.* **2022**, *7*, 9389–9404. [CrossRef]
17. Gazzola, F.; Squassina, M. Global solutions and finite time blow up for damped semilinear wave equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **2006**, *23*, 185–207. [CrossRef]
18. Wang, Y. A sufficient condition for finite time nlow up of the nonlinear Klein-Gordon equations with arbitrary positive initial energy. *Proc. AMS* **2008**, *136*, 3477–3482. [CrossRef]
19. Kafini, M. A blow-up result in a system of nonlinear viscoelastic wave equations with arbitrary positive initial energy. *Indag. Math.* **2013**, *24*, 602–612. [CrossRef]
20. Wang, Y. Arbitrary initial energy blow up for fourth-order viscous damped wave equation with exponential-type growth nonlinearity. *Appl. Math. Lett.* **2021**, *121*, 107455. [CrossRef]
21. Peng, X. Lower bounds for the blow-up time to a nonlinear viscoelastic wave equation with strong damping. *Appl. Math. Lett.* **2018**, *76*, 66–73. [CrossRef]
22. Ruf, B.; Sani, F. Ground states for elliptic equations in $\mathbb{R}^2$ with exponential critical growth. In *Geometric Properties for Parabolic and Elliptic PDE's*; Springer: Milan, Italy, 2013; pp. 251–267.
23. Ishiwata, M.; Ruf, B.; Sani, F.; Terraneo, E. Asymptotics for a parabolic equation with critical exponential nonlinearity. *J. Evol. Equ.* **2021**, *21*, 1677–1716. [CrossRef]
24. Ibrahim, S.; Jrad, R.; Majdoub, M.; Saanouni, T. Local well posedness of a 2D semilinear heat equation. *Bull. Belg. Math. Soc. Simon Stevin* **2014**, *21*, 535–551. [CrossRef]
25. Saanouni, T. A note on the inhomogeneous nonlinear heat equation in two space dimensions. *Mediterr. J. Math.* **2016**, *13*, 3651–3672. [CrossRef]
26. Ioku, N.; Ruf, B.; Terraneo, E. Existence, nonexistence, and uniqueness for a heat equation with exponential nonlinearity in $\mathbb{R}^2$. *Math. Phys. Anal. Geom.* **2015**, *18*, 29. [CrossRef]
27. Li, M.R.; Tsau, L.Y. Existence and nonexistence of global solutions of some system of semilinear wave equations. *Nonlinear Anal.* **2003**, *54*, 1397–1415. [CrossRef]
28. Souplet, P. Geometry of unbounded domains, poincare inequalities and stability in semilinear parabolic equations. *Commun. Partial Differ. Equ.* **1999**, *24*, 545–551. [CrossRef]