mathematics

Article

Conditions for Implicit-Degree Sum for Spanning Trees with Few Leaves in $K_{1,4}$-Free Graphs

Junqing Cai [1,†], Cheng-Kuan Lin [2,*], Qiang Sun [3,†] and Panpan Wang [4,†]

1 School of Mathematical Science, Tianjin Normal University, Tianjin 300387, China; caijq09@163.com
2 Department of Computer Science, National Yang Ming Chiao Tung University, Hsinchu 30010, Taiwan
3 School of Mathematical Science, Yangzhou University, Yangzhou 225009, China; qsun1987@163.com
4 School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China; wpan3746@163.com
* Correspondence: cklin@nycu.edu.tw
† These authors contributed equally to this work.

Abstract: A graph with n vertices is called an n-graph. A spanning tree with at most k leaves is referred to as a spanning k-ended tree. Spanning k-ended trees are important in various fields such as network design, graph theory, and communication networks. They provide a structured way to connect all the nodes in a network while ensuring efficient communication and minimizing unnecessary connections. In addition, they serve as fundamental components for algorithms in routing, broadcasting, and spanning tree protocols. However, determining whether a connected graph has a spanning k-ended tree or not is NP-complete. Therefore, it is important to identify sufficient conditions for the existence of such trees. The implicit-degree proposed by Zhu, Li, and Deng is an important indicator for the Hamiltonian problem and the spanning k-ended tree problem. In this article, we provide two sufficient conditions for $K_{1,4}$-free connected graphs to have spanning k-ended trees for $k = 2, 3$. We prove the following: Let G be a $K_{1,4}$-free connected n-graph. For $k = 2, 3$, if the implicit-degree sum of any $k+1$ independent vertices of G is at least $n-k+2$, then G has a spanning k-ended tree. Moreover, we give two examples to show that the lower bounds n and $n-1$ are the best possible.

Keywords: implicit-degree; spanning tree; leaves; $K_{1,4}$-free graph

MSC: 05C05; 05C07

Citation: Cai, J.; Lin, C.-K.; Sun, Q.; Wang, P. Conditions for Implicit-Degree Sum for Spanning Trees with Few Leaves in $K_{1,4}$-Free Graphs. *Mathematics* **2023**, *11*, 4981. https://doi.org/10.3390/math11244981

Academic Editors: Rasul Kochkarov, Janez Žerovnik and Andrea Scozzari

Received: 20 October 2023
Revised: 4 December 2023
Accepted: 14 December 2023
Published: 17 December 2023

1. Introduction

All the graphs considered in this paper are finite, undirected, and simple. Notations not defined here refer to [1]. For a graph G, we always use $V(G)$, $E(G)$, and $|V(G)|$ to denote the vertex set, edge set, and number of vertices of G, respectively. A graph G with n vertices is called an n-graph. Suppose H is a subgraph of G and u is a vertex of G. We define the *neighborhood* of u in H as $N_H(u) = \{x \in V(H) \mid ux \in E(G)\}$ and the *degree* of u in H as $d_H(u) = |N_H(u)|$. For two vertices $u, v \in V(H)$, we use $P_H[u,v]$ to denote a path between u and v in H with u and v as end vertices, and use $P_H(u,v) = P_H[u,v] \setminus \{u,v\}$, $d_H(u,v)$ to denote the distance between u and v in H, i.e., the number of edges of the shortest path between u and v in H. Suppose E' is a nonempty subset of $E(G)$. We use $H + E'$ and $H - E'$ to denote the graph obtained from H by adding and deleting edges in E', respectively. If $E' = \{e\}$, we write $H + e$ and $H - e$ instead of $H + \{e\}$ and $H - \{e\}$, respectively. Let $\Delta(H)$ denote the maximum degree of H. For an integer $i \geq 1$ and a subset X of $V(G)$, we define $N_H(X) = \bigcup_{x \in X} \{y \in V(H) \mid xy \in E(G)\}$, $d_H(X) = \sum_{x \in X} d_H(x)$, and $N_i^H(X) = \{x \in V(H) \mid |N_H(x) \cap X| = i\}$.

Let $G_1, G_2, \ldots, G_k$ be k graphs. The union graph of $G_1, G_2, \ldots, G_k$, denoted by $\bigcup_{i=1}^{k} G_i$, is a graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$; if $G_1, G_2, \ldots, G_k$ are pairwise vertex disjoint, we denote $\bigcup_{i=1}^{k} G_i$ by $\sum_{i=1}^{k} G_i$ or $G_1 + G_2 + \cdots + G_k$; if each G_i is isomorphic to Q, we abbreviate $\bigcup_{i=1}^{k} G_i$ as kQ. The join graph of $G_1, G_2, \ldots, G_k$, denoted by $\bigvee_{i=1}^{k} G_i$, is a graph obtained from $\sum_{i=1}^{k} G_i$ by connecting any vertex of G_i to each vertex of G_j by an edge for each $i \neq j$.

A subset U of $V(G)$ is called an independent set of G if any two vertices of U are nonadjacent in G. We use $\alpha(G)$ to denote the *independence number* of a graph G. For an integer $k \geq 1$, we denote $\sigma_k(G) = \min\{\sum_{i=1}^{k} d_G(u_i) \mid \{u_1, u_2, \ldots, u_k\}$ as an independent set of $G\}$ if $k \leq \alpha(G)$; otherwise, $\sigma_k(G) = +\infty$.

A tree is a connected acyclic graph. A maximal tree is a tree that cannot be extended by adding any more edges without creating a cycle. For a tree T, a *leaf* of T is a vertex v with $d_T(v) = 1$. We use $L(T)$ to denote the set of leaves in T. A *spanning tree* (resp. *a Hamiltonian path*) of a graph is a tree (resp. a path) containing all the vertices of the graph. A *spanning k-ended tree* (resp. *k-ended tree*) is a spanning tree (resp. a tree) with at most k leaves. Obviously, a Hamiltonian path is a spanning 2-ended tree.

Spanning k-ended trees are important in various fields such as network design, graph theory, and communication networks. They provide a structured way to connect all the nodes in a network while ensuring efficient communication and minimizing unnecessary connections. In addition, they serve as fundamental components for algorithms in routing, broadcasting, and spanning tree protocols. Therefore, the existence and properties of spanning k-ended trees are crucial for optimizing network design and performance. Ozeki and Yamashita [2] pointed out that determining whether a graph has a spanning k-ended tree or not is NP-complete. Since then, many scholars have studied the sufficient conditions for the existence of spanning k-ended trees, such as degree sum conditions [3–9].

The forbidden induced subgraph conditions are a set of criteria used to determine whether a given graph can have a spanning tree with specific properties. These conditions indicate which specific subgraphs are not allowed to be induced in the graph for such spanning trees to exist. Among all the forbidden induced subgraphs, the complete bipartite graph $K_{1,r}$ is central. A graph that does not contain an induced subgraph isomorphic to $K_{1,r}$ is called a $K_{1,r}$*-free* graph. Matthews and Sumner [10] showed that a $K_{1,3}$-free n-graph G has a Hamiltonian path if $\sigma_3(G) \geq n - 2$. Kano et al. [11] gave a degree sum condition for a $K_{1,3}$-free graph to have a spanning k-ended tree as follows.

Theorem 1 (Kano et al. [11])**.** *Let G be a connected $K_{1,3}$-free n-graph. If $\sigma_{k+1}(G) \geq n - k$ for any $k \geq 2$, then G has a spanning k-ended tree.*

For $k = 2, 3$, Kyaw gave a sufficient condition for a $K_{1,4}$-free graph to have a spanning k-ended tree.

Theorem 2. *Let G be a connected $K_{1,4}$-free n-graph.*

(1) (Kyaw [12]) *If $\sigma_3(G) \geq n$, then G has a Hamiltonian path.*
(2) (Kyaw [13]) *If $\sigma_4(G) \geq n - 1$, then G has a spanning 3-ended tree.*

The Hamiltonian problem holds significant importance in graph theory and combinatorial optimization. It is crucial in determining whether a given graph contains a Hamiltonian cycle, which is a cycle that visits each vertex exactly once. The problem has applications in various fields, including computer science, logistics, and transportation, as it relates to the design of efficient routes, scheduling, and circuit layout. Furthermore, the study of the Hamiltonian problem has led to the development of important algorithms and heuristics, contributing to advancements in computational complexity and theoretical computer science. The problem also serves as a foundational concept for understanding and solving other NP-complete problems, making it a central focus of research in combinatorial optimization and algorithmic design.

Degree conditions and forbidden induced subgraph conditions are the two types of classical sufficient conditions for graphs to be Hamiltonian. As we all known, in the study of the existence of a Hamiltonian cycle, the degree sum of end vertices in a longest path is crucial. Bondy [14] proved that if a 2-connected n-graph has a longest path between x and y such that $d_G(x) + d_G(y) \geq c$, then G is Hamiltonian or has a cycle of length at least c. Sometimes, perhaps, the degree sum of the two end vertices of a longest path is smaller, but the degrees of their neighbors or vertices at distance two with them are larger, and we can replace them by some larger degree vertices in the right position so that we can construct a longest path with a larger degree sum of its end vertices. Therefore, we can construct a longer cycle. With the inspiration of this idea, Zhu, Li, and Deng [15] proposed the definition of implicit-degree.

Definition 1 (Zhu, Li and Deng [15])**.** *Let x be a vertex of G and let $N_G^2(x) = \{y \in V(G) \mid d_G(x,y) = 2\}$ denote the set of vertices at distance two with x in G. Set $M_2 = \max\{d_G(y) \mid y \in N_G^2(x)\}$. If $l = d_G(x) \geq 2$ and $N_G^2(x) \neq \emptyset$, then suppose that $d_1^x \leq d_2^x \leq \ldots \leq d_{l-1}^x \leq d_l^x \leq \ldots$ is the degree sequence of vertices of $N_G(x) \cup N_G^2(x)$ in G. The* implicit-degree *of x, denoted by $id_G(x)$, is defined as*

$$id_G(x) = \begin{cases} \max\{d_l^x, d_G(x)\}, & \text{if } d_l^x > M_2; \\ \max\{d_{l-1}^x, d_G(x)\}, & \text{if } d_l^x \leq M_2. \end{cases}$$

If $l = d_G(x) \leq 1$ or $N_G^2(x) = \emptyset$, then we define $id_G(x) = d_G(x)$.

Obviously, $id_G(x) \geq d_G(x)$ for every vertex $x \in V(G)$. From the definition of implicit-degree, it coincides with the importance of a person's friends or friends' friends in a social network. We define $i\sigma_k(G) = \min\{\sum_{j=1}^{k} id_G(u_j) \mid \{u_1, u_2, \ldots, u_k\}$ is an independent set of $G\}$ if $k \leq \alpha(G)$; otherwise, $i\sigma_k(G) = +\infty$. Many classical results in graph theory that consider degree conditions can be extended to implicit-degree conditions, such as [15–17]. We just give one example related to spanning trees to show this in detail.

Theorem 3 (Cai et al. [18])**.** *Let G be a connected n-graph and $k \geq 2$ be an integer. If $i\sigma_t(G) \geq \frac{t(n-k)}{2} + 1$ $(k \geq t \geq 2)$, then G contains a spanning k-ended tree.*

Since $i\sigma_2(G) \geq \sigma_2(G)$, the result of Broersma and Tuinstra [3] is a corollary of Theorem 3 when $t = 2$. In this article, we extend Theorem 2 by using $i\sigma_3(G)$ and $i\sigma_4(G)$ in place of $\sigma_3(G)$ and $\sigma_4(G)$, respectively.

Theorem 4. *Suppose G is a connected $K_{1,4}$-free n-graph.*

(1) If $i\sigma_3(G) \geq n$, then G has a Hamiltonian path.
(2) If $i\sigma_4(G) \geq n - 1$, then G has a spanning 3-ended tree.

The proof of Theorem 4 will be given in Section 3. Now, we present the following three examples. The first one shows that the lower bounds in Theorem 4 are better than those in Theorem 3; the second one shows that the lower bounds are sharp in Theorem 4; and the third one provides graphs which do not satisfy the conditions of Theorem 2, but satisfy the conditions of Theorem 3.

Example 1. *Let G be a connected n-graph with $n > 6$. Theorem 3 shows that (1) G has a Hamiltonian path if $i\sigma_3(G) \geq (3n-4)/2 > n$; (2) G has a spanning 4-ended tree if $i\sigma_4(G) \geq 2n-7 > n-1$. Theorem 4 shows that, if G is $K_{1,4}$-free, then $i\sigma_3(G) \geq (3n-4)/2$ and $i\sigma_4(G) \geq 2n-7$ can be reduced to $i\sigma_3(G) \geq n$ and $i\sigma_4(G) \geq n-1$, respectively.*

Example 2. *(1) The graph $G = K_1 \vee 3K_m$ indicates that the condition $i\sigma_3 \geq n(= 3m+1)$ in Theorem 4 is sharp. Clearly, G has no Hamiltonian path. The vertex in K_1 has implicit-degree $3m$ and every vertex in $3K_m$ has implicit-degree m. So, $i\sigma_3(G) = 3m = n-1$. (2) The following graph G indicates that the condition $i\sigma_4(G) \geq n-1$ in Theorem 4 is sharp. Let G_i be a complete n_i-graph for $1 \leq i \leq 4$. The graph G is constructed with vertex set $V(G) = \bigcup_{i=1}^{4} V(G_i) \cup \{u, v\}$ ($u, v \notin \bigcup_{i=1}^{4} V(G_i)$) and edge set $E(G) = \bigcup_{i=1}^{4} E(G_i) \cup \{uv\} \cup \{uw \mid w \in V(G_1) \cup V(G_2)\} \cup \{vw \mid w \in V(G_3) \cup V(G_4)\}$ (see Figure 1). It is easy to verify that $i\sigma_4(G) = \sum_{i=1}^{4} n_i = n-2$ and G has no spanning 3-ended tree. Without loss of generality, suppose $n_1 + n_2 \leq n_3 + n_4$. For each vertex $x \in V(G)$, the degree and implicit-degree of x can be seen from in Table 1.*

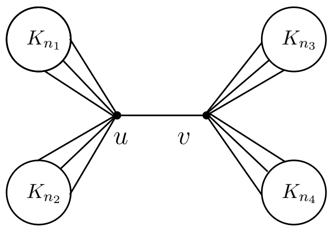

Figure 1. Graph with no spanning 3-ended tree.

Table 1. Degrees and implicit-degrees of all the vertices in the graph in Figure 1.

The Vertex x	$N_G(x)$	$N_G^2(x)$	$d_G(x)$	$id_G(x)$
$x \in V(K_{n_1})$	$(V(K_{n_1}) \setminus \{x\}) \cup \{u\}$	$V(K_{n_2}) \cup \{v\}$	n_1	n_1
$x \in V(K_{n_2})$	$(V(K_{n_2}) \setminus \{x\}) \cup \{u\}$	$V(K_{n_1}) \cup \{v\}$	n_2	n_2
$x \in V(K_{n_3})$	$(V(K_{n_3}) \setminus \{x\}) \cup \{v\}$	$V(K_{n_4}) \cup \{u\}$	n_3	n_3
$x \in V(K_{n_4})$	$(V(K_{n_4}) \setminus \{x\}) \cup \{v\}$	$V(K_{n_3}) \cup \{u\}$	n_4	n_4
$x = u$	$V(K_{n_1}) \cup V(K_{n_2}) \cup \{v\}$	$V(K_{n_3}) \cup V(K_{n_4})$	$n_1 + n_2 + 1$	$n_3 + n_4 + 1$
$x = v$	$V(K_{n_3}) \cup V(K_{n_4}) \cup \{u\}$	$V(K_{n_1}) \cup V(K_{n_2})$	$n_3 + n_4 + 1$	$n_3 + n_4 + 1$

Example 3. *(1) Let $G_i = K_m$ with $i = 1, 2, 3$. Let G' be a graph constructed from the graph $K_1 \vee (G_1 \cup G_2 \cup G_3)$ ($m \geq 2$) by adding one edge between x_i and y_i for each $1 \leq i \leq m$, and deleting one edge between K_1 and G_1, G_2. Where $V(K_1) = \{u\}$, $V(G_1) = \{x_1, x_2, \ldots, x_m\}$, $V(G_2) = \{y_1, y_2, \ldots, y_m\}$, and $V(G_3) = \{z_1, z_2, \ldots, z_m\}$, and x_1 and y_1 denote the two vertices not adjacent to u. It is easy to verify that $\sigma_3(G') = 3m = n-1$, $i\sigma_3(G') = 3(m+1) > n$, and G' has a Hamiltonian path $x_m x_{m-1} x_1 y_1 y_2 \ldots y_m x z_1 z_2 \ldots z_m$. For each vertex $x \in V(G')$, the degree and implicit-degree of x can be seen from Table 2. (2) Let G'' be a graph constructed from the graph in Figure 1 with $n_i = m = (n-2)/4$ ($1 \leq i \leq 4$) by adding $m-1$ independent edges between G_1 and G_2 (see Figure 2). It is easy to verify that $\sigma_4(G'') = 4m = n-2$, $i\sigma_4(G'') = 2m + 2(m+1) > n-1$, and G'' contains a spanning 3-ended tree. This shows that Theorem 4 generalizes Theorem 2.*

Table 2. Degrees and implicit-degrees of all the vertices in the graph G'.

The Vertex x	$N_{G'}(x)$	$N^2_{G'}(x)$	$d_{G'}(x)$	$id_{G'}(x)$
$x = u$	$V(G) \setminus \{u\}$	$\emptyset$	$3m$	$3m$
$x = x_i (2 \leq i \leq m)$	$(V(G_1) \setminus \{x_i\}) \cup \{x, y_i\}$	$(V(G_2) \setminus \{y_i\}) \cup V(G_3)$	$m+1$	$m+1$
$x = x_1$	$(V(G_1) \setminus \{x_1\}) \cup \{y_1\}$	$(V(G_2) \setminus \{y_1\}) \cup \{x\}$	m	$m+1$
$x = y_i (2 \leq i \leq m)$	$(V(G_2) \setminus \{y_i\}) \cup \{x, x_i\}$	$(V(G_1) \setminus \{x_i\}) \cup V(G_3)$	$m+1$	$m+1$
$x = y_1$	$(V(G_2) \setminus \{y_1\}) \cup \{x_1\}$	$(V(G_1) \setminus \{x_1\}) \cup \{x\}$	m	$m+1$
$x \in z_i (1 \leq i \leq m)$	$(V(G_3) \setminus \{z_i\}) \cup \{x\}$	$(V(G_1) \setminus \{x_1\}) \cup (V(G_2) \setminus \{y_1\})$	m	$m+1$

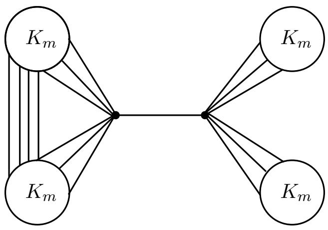

Figure 2. Graph with no Hamiltonian cycle.

We will prove Theorem 4 in Section 3 while some preliminaries will be given in Section 2.

2. Preliminaries

Let x, y be two vertices of an oriented path P. We use xPy to denote the subpath of P from x to y and $y\overline{P}x$ to denote the subpath of P from y to x in the reverse direction. Define x^- and x^+ as the predecessor and successor of x on P, respectively. For any subset $I \subseteq V(P)$, we define $I^- = \{y \mid y^+ \in I\}$ and $I^+ = \{y \mid y^- \in I\}$. In this section, let $P = x_1 x_2 \ldots x_p$ be a path of a connected graph G and x, y, z be any three distinct vertices not in $V(P)$. The following lemmas are useful in the proof of Theorem 4.

Lemma 1 (Kyaw [13]). *Let T be a maximal tree of G with four leaves. If G has no spanning 3-ended tree, then there is no 3-ended tree T' in G such that $V(T') = V(T)$.*

Lemma 2 (Zhu, Li and Deng [15]). *If P is a longest path satisfying $x_1 x_p \notin E(G)$ and $d_G(x_1) < id_G(x_1)$, then there is a vertex $x_j \in N_P(x_1)^-$ such that $d_G(x_j) \geq id_G(x_1)$.*

Lemma 3. *If $N_P(x)^- \cap N_P(y) = \emptyset$, then*

$$d_P(x) + d_P(y) \leq \begin{cases} |V(P)|, & \text{if } xx_1 \notin E(G) \text{ or } yx_p \notin E(G); \\ |V(P)| + 1, & \text{otherwise.} \end{cases}$$

Proof. Note that $|N_P(x)^- \cup N_P(y)| = |N_P(x)^-| + |N_P(y)|$. If $xx_1 \notin E(G)$, then $N_P(x)^- \cup N_P(y) \subseteq V(P)$ and thus $d_P(x) + d_P(y) = |N_P(x)^-| + |N_P(y)| = |N_P(x)^- \cup N_P(y)| \leq |V(P)|$. If $yx_p \notin E(G)$, then $(N_P(x) \setminus \{x_1\})^- \cup N_P(y) \subseteq V(P) \setminus \{x_p\}$ and thus $d_P(x) + d_P(y) = |N_P(x)^-| + |N_P(y)| = |N_P(x)^- \cup N_P(y)| \leq |V(P)|$. Otherwise, $(N_P(x) \setminus \{x_1\})^- \cup N_P(y) \subseteq V(P)$ and thus $d_P(x) + d_P(y) = |(N_P(x) \setminus \{x_1\})^-| + |\{x_1\}| + |N_P(y)| = |(N_P(x) \setminus \{x_1\})^- \cup N_P(y)| + 1 \leq |V(P)| + 1$. □

Lemma 4. *If $N_P(x)^- \cap N_P(y) = \emptyset$, $N_P(x)^- \cap N_P(z) = \emptyset$, and $N_P(y) \cap N_P(z) = \emptyset$, then $d_P(x) + d_P(y) + d_P(z) \leq |V(P)| + 1$.*

Proof. Note that $|N_P(x)^- \cup N_P(y) \cup N_P(z)| = |N_P(x)^-| + |N_P(y)| + |N_P(z)|$ and $(N_P(x) \setminus \{x_1\})^- \cup N_P(x) \cup N_P(z) \subseteq V(P)$. Then, $d_P(x) + d_P(y) + d_P(z) \leq |(N_P(x) \setminus \{x_1\})^-| + |\{x_1\}| + |N_P(y)| + |N_P(z)| \leq |V(P)| + 1$. □

3. Proof of Theorem 4

Since the result of Theorem 4 (2) will be used in the proof of Theorem 4 (1), we prove Theorem 4 (2) firstly.

Proof of Theorem 4 (2). Let G be a connected $K_{1,4}$-free n-graph with $i\sigma_4(G) \geq n-1$ and let G have no spanning three-ended tree. Then, every spanning tree of G has at least four leaves. Choose a maximal tree T of G with exactly four leaves such that $\Delta(T)$ is minimal.

Since T has exactly four leaves, we have $\Delta(T) = 4$ or $\Delta(T) = 3$ (see Figure 3). Let $L(T) = \{x_1, x_2, x_3, x_4\}$ and $S = \{x \mid x \in V(T) \text{ and } d_T(x) = \Delta(T)\}$. Since T is maximal, $N_G(L(T)) \subseteq V(T)$.

If $\Delta(T) = 4$, then $|S| = 1$. Let $S = \{r\}$ and $N_T(r) = \{y_1, y_2, y_3, y_4\}$ (see Figure 3a). If $y_iy_j \in E(G)$ for some $i \neq j$, then $T' = T + y_iy_j - ry_i$ is a tree with four leaves such that $V(T') = V(T)$ and $\Delta(T') = 3 < \Delta(T)$. This contradicts the choice of T. So, $\{y_1, y_2, y_3, y_4\}$ is an independent set of G. Thus, $\{r, y_1, y_2, y_3, y_4\}$ induces a $K_{1,4}$, a contradiction.

Next, we can assume $\Delta(T) = 3$. Then, $|S| = 2$. Let $S = \{s, t\}$ (see Figure 3b). We choose such a T satisfying the following conditions.

(C1) The distance $d_T(s,t)$ is as small as possible;

(C2) The degree sum $d_G(L(T))$ is as large as possible, subject to (1).

Let B_i be the component of $T - S$ such that $V(B_i) \cap L(T) = \{x_i\}$ and let y_i be the unique vertex of $N_T(S) \cap V(B_i)$ with $1 \leq i \leq 4$. We assume, without loss of generality, that $sy_1, sy_2, ty_3, ty_4 \in E(T)$. □

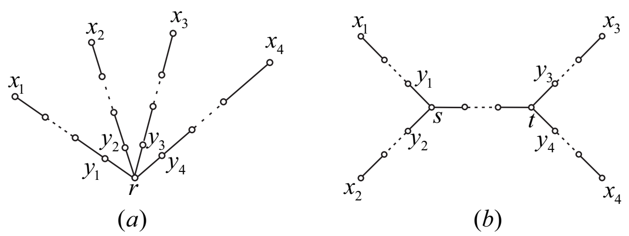

Figure 3. Trees with exactly four leaves.

Claim 1. *$L(T)$ is an independent set of G.*

Proof. If there are two vertices $x_i, x_j \in L(T)$ such that $x_ix_j \in E(G)$, then $T' = T + x_ix_j - sy_i$ or $T + x_ix_j - ty_i$ is a three-ended tree such that $V(T') = V(T)$, contrary to Lemma 1. □

Claim 2. *$x_iy_j \notin E(G)$ for $1 \leq i \neq j \leq 4$.*

Proof. If there are two vertices x_i, y_j such that $x_iy_j \in E(G)$, then $T' = T + x_iy_j - sy_j$ or $T + x_iy_j - ty_j$ is a three-ended tree such that $V(T') = V(T)$, contrary to Lemma 1. □

Claim 3. *$N_G(L(T)) \cap V(P_T(s,t)) = \emptyset$.*

Proof. Suppose to the contrary that $N_G(L(T)) \cap V(P_T(s,t)) \neq \emptyset$. We assume, without loss of generality, that there is a vertex $z \in V(P_T(s,t))$ such that $x_1z \in E(G)$. Then, $T' = T + x_1z - sy_1$ is a tree with four leaves such that $\Delta(T') = 3$, $V(T') = V(T)$ and $d_{T'}(z) = d_{T'}(t) = 3$. But $d_{T'}(z,t) < d_T(s,t)$, contrary to the condition (C1). □

Claim 4. *$id_G(x_j) = d_G(x_j)$ for each vertex $x_j \in L(T)$.*

Proof. Suppose that there is a vertex $x_j \in L(T)$ such that $id_G(x_j) > d_G(x_j)$. Without loss of generality, we assume $id_G(x_1) > d_G(x_1)$. Let $l = d_G(x_1)$ and $N_G(x_1) = \{w_1, w_2, \ldots, w_l\}$. Suppose that $d_1^{x_1} \leq d_2^{x_1} \leq \ldots \leq d_{l-1}^{x_1} \leq d_l^{x_1} \leq \ldots$ is the degree sequence of vertices of $N_G(x_1) \cup N_G^2(x_1)$ in G. By the definition of $id_G(x_1)$, we have $l \geq 2$ and $id_G(x_1) = d_{l-1}^{x_1}$

or $id_G(x_1) = d_l^{x_1}$. Denote w_i^- and w_i^+ as the predecessor and successor of w_i on the path $P_T[x_1, w_i]$, respectively. Without loss of generality, suppose $w_1^- = x_1$. By Claims 1–3, there must exist a vertex $w_m^+ \in N_G^2(x_1)$ for some $1 \leq m \leq l$ and $\{s,t\} \cap \{w_1^-, w_2^-, \ldots, w_l^-\} = \emptyset$.

If $id_G(x_1) = d_{l-1}^{x_1}$, then, since $w_2^-, w_3^-, \ldots, w_l^-$ are $l-1$ vertices in $N_G(x_1) \cup N_G^2(x_1)$, there must exist a vertex $w_j^- \in \{w_2^-, w_3^-, \ldots, w_l^-\}$ such that $d_G(w_j^-) \geq d_{l-1}^{x_1} = id_G(x_1) > d_G(x_1)$. If $id_G(x_1) = d_l^{x_1}$, then $d_l^{x_1} > M_2 = \max\{d_G(xu) \mid u \in N_G^2(x_1)\}$. Since $w_m^+ \in N_G^2(x_1)$, $d_G(w_m^+) \leq M_2 < d_l^{x_1} = id_G(x_1)$. Since $w_2^-, w_3^-, \ldots, w_l^-, w_m^+$ are l vertices in $N_G(x_1) \cup N_G^2(x_1)$, there must exist a vertex $w_j^- \in \{w_2^-, w_3^-, \ldots, w_l^-\}$ such that $d_G(w_j^-) \geq d_l^{x_1} = id_G(x_1) > d_G(x_1)$. Therefore, in both cases, we can obtain a tree $T' = T + x_1w_j - w_jw_j^-$ with four leaves such that $\Delta(T') = \Delta(T)$, $V(T') = V(T)$, and w_j^- replaces x_1 as a new leaf of T'. Then, $\sum\limits_{u \in L(T')} d_G(u) - \sum\limits_{u \in L(T)} d_G(u) = d_G(w_j^-) - d_G(x_1) \geq id_G(x_1) - d_G(x_1) > 0$, contrary to condition (C2). □

Claim 5. *For $1 \leq i \neq j \leq 4$, if $z \in V(B_i) \cap N_G(x_j)$, then $z^- \notin N_G(L(T) \setminus \{x_j\})$, where z^- denotes the predecessor of z on the path $P_T[x_j, z]$.*

Proof. Suppose z is a vertex of $V(B_i) \cap N(x_j)$ such that $z^- \notin N_G(L(T) \setminus \{x_j\})$ for some $i \neq j$. Without loss of generality, we assume that $sy_i \in E(G)$ and $z^-x_k \in E(G)$ for some $k \neq j$. Then, $T' = T + \{x_jz, x_kz^-\} - \{sy_i, zz^-\}$ is a three-ended tree such that $V(T') = V(T)$, contrary to Lemma 1. □

Claim 6. *$N_4(L(T) = N_3(L(T)) = \emptyset$.*

Proof. If there is a vertex $w \in N_4(L(T))$, then, by Claim 1, $\{w, x_1, x_2, x_3, x_4\}$ induces a $K_{1,4}$, a contradiction.

If there is a vertex $w \in N_3(L(T))$, then, by Claim 3, $w \in \bigcup\limits_{i=1}^{4} V(B_i) \cup S$. Let $wx_r, wx_s, wx_t \in E(G)$, where $\{x_r, x_s, x_t\} \subset L(T)$. We can assume $r \neq i$. If $w \in V(B_i)$ for some $1 \leq i \leq 4$, then $w \neq y_i$ by Claim 2 and $w^- \notin N_G(L(T))$ by Claim 5. Thus, $\{w, w^-, x_r, x_s, x_t\}$ induces a $K_{1,4}$ by Claim 1, a contradiction.

If $w \in S$, then, without loss of generality, we assume $w = s$. So there is a vertex $x_j \in L(T)$ such that $sx_j \in E(G)$ and $t \in V(P_T[s, x_j])$. If st is an edge of T, then $T' = T + sx_j - st$ is a three-ended tree such that $V(T') = V(T)$, contrary to Lemma 1. Otherwise, there is a vertex $s^+ \in V(P_T[s,t])$. By Claim 3, $x^+ \notin N_G(L(T))$. Then, $\{s, s^+, x_r, x_s, x_t\}$ induces a $K_{1,4}$ by Claim 1, a contradiction. □

Next, we calculate $d_G(L(T))$. For convenience, for $1 \leq i \leq 4$, we set $A_i^1 = \{x_i\}$, $A_i^2 = N_G(x_i) \cap V(B_i)$, $A_i^3 = (N_G(L(T) \setminus \{x_i\}))^- \cap V(B_i)$, and $A_i^4 = (N_2^G(L(T)) \setminus N(x_i)) \cap V(B_i)$, where $(N(L(T) \setminus \{x_i\}))^- = \{u^- \mid u \in N(L(T) \setminus \{x_i\})\}$. Clearly, $A_i^1 \cup A_i^2 \cup A_i^3 \cup A_i^4 \subseteq V(B_i)$ for every $i = 1,2,3,4$.

Claim 7. *For $1 \leq i \leq 4$, A_i^1, A_i^2, A_i^3, and A_i^4 are pairwise disjoint.*

Proof. Clearly, $A_i^1 \cap A_i^2 = \emptyset$ and $A_i^1 \cap A_i^3 = \emptyset$. By Claim 1, $A_i^1 \cap A_i^4 = \emptyset$. By Claim 5, $A_i^2 \cap A_i^3 = \emptyset$ and $A_i^3 \cap A_i^4 = \emptyset$. By Claim 6, $A_i^2 \cap A_i^4 = \emptyset$. □

For $1 \le i \le 4$, by Claims 6 and 7, and the inclusion–exclusion principle, we have

$$\begin{aligned} |V(B_i)| &\ge |A_i^1 \cup A_i^2 \cup A_i^3 \cup A_i^4| \\ &= |A_i^1| + |A_i^2| + |A_i^3| + |A_i^4| \\ &= 1 + |N_G(x_i) \cap V(B_i)| + |(N_G(L(T) \setminus \{x_i\}))^- \cap V(B_i)| \\ &\quad + |(N_2^G(L(T)) \setminus N(x_i)) \cap V(B_i)| \\ &= 1 + |N_G(x_i) \cap V(B_i)| + |(N_G(L(T) \setminus \{x_i\})) \cap V(B_i)| \\ &\quad + |(N_2^G(L(T)) \setminus N(x_i)) \cap V(B_i)| \\ &\ge 1 + \sum_{j=1}^{4} |N_G(x_j) \cap V(B_i)|. \end{aligned}$$

Hence,

$$\sum_{i=1}^{4} |V(B_i)| \ge 4 + \sum_{i=1}^{4} \sum_{j=1}^{4} |N_G(x_j) \cap V(B_i)|. \quad (1)$$

By Claim 6, we have

$$\sum_{j=1}^{4} |N_G(x_j) \cap \{s\}| \le 2 \text{ and } \sum_{j=1}^{4} |N_G(x_j) \cap \{t\}| \le 2. \quad (2)$$

Notice that $N_G(L(T)) \subseteq V(T)$ and $N_G(L(T)) \cap (V(P_T[s,t]) \setminus \{s,t\}) = \emptyset$ by Claim 3. By inequalities (1) and (2), we have

$$\begin{aligned} \sum_{j=1}^{4} d_G(x_j) &= \sum_{i=1}^{4} \sum_{j=1}^{4} |N_G(x_j) \cap V(B_i)| + \sum_{j=1}^{4} |N_G(x_j) \cap \{s,t\}| \\ &\le (\sum_{i=1}^{4} |V(B_i)| - 4) + 4 \\ &= \sum_{i=1}^{4} |V(B_i)| \le |V(T)| - 2. \end{aligned}$$

Therefore, by Claim 4, we have $n = |V(G)| \ge |V(T)| \ge 2 + \sum_{j=1}^{4} d_G(x_j) = 2 + \sum_{j=1}^{4} id_G(x_j) \ge 2 + i\sigma_4(G)$. So, $i\sigma_4(G) \le n - 2$; this contradicts the condition $i\sigma_4(G) \ge n - 1$. Now, we complete the proof of Theorem 4 (2).

□

Proof of Theorem 4 (1). Let G be a connected $K_{1,4}$-free n-graph with $i\sigma_3(G) \ge n$ but let G have no Hamiltonian path. Since $i\sigma_4(G) \ge i\sigma_3(G) \ge n$, by Theorem 4 (2), G has a spanning three-ended tree. Let

$$\mathcal{T}^* = \{T \mid T \text{ is a spanning three-ended tree of } G\}.$$

Since G has no Hamiltonian path, every tree in $\mathcal{T}^*$ has exactly three leaves. We choose a longest path $P = x_1x_2 \ldots x_p$ of $\mathcal{T}^*$ such that $d_G(x_1) + d_G(x_p)$ is as large as possible. Then, $N_G(x_1) \cup N_G(x_p) \subseteq V(P)$. Let T be the tree in $\mathcal{T}^*$ containing the path P. Then, $H = T - V(P)$ is a path of T and one of the end vertices of H is adjacent to some vertex x_r with $2 \le r \le p-1$ in T. For convenience, set $H = y_1y_2 \ldots y_h$. Without loss of generality, we assume $y_1x_r \in E(T)$. In fact, by the choice of P, we have $h + 1 \le r \le p - h$. □

Claim 8. *$\{x_1, x_p, y_j\}$ is an independent set of G for every vertex $y_j \in V(H)$.*

Proof. If $x_1x_p \in E(G)$, then $y_h\overline{H}y_1x_r\overline{P}x_1x_p\overline{P}x_{r+1}$ is a Hamiltonian path of G, a contradiction. Since $N_G(x_1) \cup N_G(x_p) \subseteq V(P)$, we have $x_1y_j \notin E(G)$ and $x_py_j \notin E(G)$ for every vertex $y_j \in V(H)$. So, $\{x_1, x_p, y_j\}$ is an independent set of G for every vertex $y_j \in V(H)$. $\square$

Claim 9. *$id_G(x_1) = d_G(x_1)$ and $id_G(x_p) = d_G(x_p)$.*

Proof. If $d_G(x_1) < id_G(x_1)$, then, by Lemma 2, there is a vertex $x_j \in N_P(x_1)^-$ such that $d_G(x_j) \geq id_G(x_1)$. Thus, $P' = x_j\overline{P}x_1x_{j+1}Px_p$ is another longest path of $\mathcal{T}^*$ such that $d_G(x_j) + d_G(x_p) \geq id_G(x_1) + d_G(x_p) > d_G(x_1) + d_G(x_p)$. This contradicts the choice of P. So, $id_G(x_1) = d_G(x_1)$. Similarly, $id_G(x_p) = d_G(x_p)$. $\square$

Set $i\delta(H) = \min\{id_G(u) \mid u \in V(H)\}$.

Case 1. *There is a vertex $y_t \in V(H)$ such that $d_G(y_t) \geq i\delta(H)$.*

Subcase 1. $|V(H)| \geq 2$.

Set $A_1 = N_P(x_1)^-$, $B_1 = N_P(x_p)^+$ and $C_1 = N_P(y_t)$. If there is a vertex $x_s \in A_1 \cap C_1$, then $T' = T + \{y_tx_s, x_1x_{s+1}\} - \{x_sx_{s+1}, y_1x_r\}$ is a tree in $\mathcal{T}^*$ and $P' = y_1Hy_tx_s\overline{P}x_1x_{s+1}Px_p$ is a path of $\mathcal{T}^*$ longer than P, a contradiction. So, $A_1 \cap C_1 = \emptyset$. Similarly, $B_1 \cap C_1 = \emptyset$.

If there is a vertex $x_s \in A_1 \cap B_1$, then $s \neq r$ (otherwise, $y_h\overline{H}y_1x_r\overline{P}x_1x_{r+1}Px_p$ is a Hamiltonian path of G, a contradiction). Thus, $T' = T + \{x_1x_{s+1}, x_px_{s-1}\} - \{x_rx_{r-1}, x_sx_{s-1}\}$ is a tree in $\mathcal{T}^*$, and $P' = y_h\overline{H}y_1x_rPx_px_{s-1}\overline{P}x_1x_{s+1}Px_{r-1}$ if $s < r$ or $P' = y_h\overline{H}y_1x_rPx_{s-1}x_p\overline{P}x_{s+1}x_1Px_{r-1}$ if $s > r$; P' is a path of $\mathcal{T}^*$ longer than P, a contradiction. So, $A_1 \cap B_1 = \emptyset$. Therefore, A_1, B_1, and C_1 are pairwise disjoint. Note that $A_1 \cup B_1 \cup C_1 \subseteq V(P)$ and $N_G(x_1) \cup N_G(x_p) \subseteq V(P)$. We have

$$\begin{aligned} d_G(x_1) + d_G(x_p) + d_G(y_t) &= (d_P(x_1) + d_P(x_p) + d_P(y_t)) + (d_H(x_1) + d_H(x_p) + d_H(y_t)) \\ &\leq (|A_1| + |B_1| + |C_1|) + (|V(H)| - 1) \\ &\leq |V(P)| + (|V(H)| - 1) \\ &= n - 1. \end{aligned}$$

On the other hand, $d_G(x_1) + d_G(x_p) + d_G(y_t) \geq id_G(x_1) + id_G(x_p) + i\delta(H) \geq i\sigma_3(G) \geq n$, a contradiction.

Subcase 2. $|V(H)| = 1$.

Then, $H = \{y_1\}$. Let $N_P(y_1) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ $(k \geq 1)$. We can assume that $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ occur in this order along P. Let $P_1 = x_2x_3\ldots x_{i_1}$, $P_j = x_{i_{j-1}+1}x_{i_{j-1}+2}\ldots x_{i_j}$ for $2 \leq j \leq k$, and, $P_{k+1} = x_{i_k+1}x_{i_k+2}\ldots x_{p-1}$. Clearly, $n = \sum_{j=1}^{k+1}|V(P_j)| + 3$. Since P is a longest path of $\mathcal{T}^*$, it is easy to verify that $N_{P_s}(x_1)^- \cap N_{P_s}(x_p) = \emptyset$ for $1 \leq s \leq k+1$, $x_{i_j+1} \notin N_P(x_1)$, $x_{i_j-1} \notin N_P(x_p)$ for $1 \leq j \leq k$, and moreover, if $k \geq 2$, then $x_{i_j+1} \notin N_P(x_p)$ for $1 \leq j \leq k-1$, $x_{i_j-1} \notin N_P(x_1)$ and $i_j - i_{j-1} \geq 2$ for $2 \leq j \leq k$.

If $k = 1$, then, by Lemma 3, $d_G(x_1) + d_G(x_p) + d_G(y_1) = d_P(x_1) + d_P(x_p) + d_P(y_1) = \sum_{j=1}^{2}(d_{P_j}(x_1) + d_{P_j}(x_p)) + 1 \leq \sum_{j=1}^{2}(|V(P_j)| + 1) + 1 = n - 1$.

If $k = 2$, then $x_{i_1} \notin N_P(x_p)$ (otherwise, $\{x_{i_1}, x_{i_1-1}, x_{i_1+1}, y_1, x_p\}$ induces a $K_{1,4}$, a contradiction). Thus, by Lemma 3, we have $d_G(x_1) + d_G(x_p) + d_G(y_1) = d_P(x_1) + d_P(x_p) + d_P(y_1) = \sum_{j=1}^{3}(d_{P_j}(x_1) + d_{P_j}(x_p)) + 2 \leq \sum_{j=1}^{3}|V(P_j)| + 2 = n - 1$.

Next, suppose $k \geq 3$. Similarly to the case $k = 2$, we can ascertain that $d_{P_j}(x) + d_{P_j}(y) \leq |V(P_j)|$ for $j = 1, k, k+1$. Since G is $K_{1,4}$-free, $\{x_{i_{j-1}+1}, x_{i_j-1}, x_{i_j}\} \cap (N_P(x_1) \cup N_P(x_p)) = \emptyset$ for $2 \leq j \leq k-1$. By Lemma 3, we have $d_{P_j}(x_1) + d_{P_j}(x_p) = d_{P_j \setminus \{x_{i_{j-1}}, x_{i_j}\}}(x_1) + d_{P_j \setminus \{x_{i_{j-1}}, x_{i_j}\}}(x_p) \leq |V(P_j \setminus \{x_{i_{j-1}}, x_{i_j}\})| = |V(P_j)| - 2$ for $2 \leq j \leq k-1$. Therefore, $d_G(x_1) + d_G(x_p) + d_G(y_1) = d_P(x_1) + d_P(x_p) + d_P(y_1) = \sum_{j=1}^{k+1}(d_{P_j}(x_1) + d_{P_j}(x_p)) + k \leq |V(P_1)| + \sum_{j=2}^{k-1}(|V(P_j)| - 2) + |V(P_k)| + |V(P_{k+1})| + k \leq n-1$.

By the above discussion, for any $k \geq 1$, we have $d_G(x_1) + d_G(x_p) + d_G(y_1) \leq n-1$. On the other hand, $d_G(x_1) + d_G(x_p) + d_G(y_1) \geq id_G(x_1) + id_G(x_p) + i\delta(H) \geq i\sigma_3(G) \geq n$, a contradiction.

Case 2. $d_G(y_j) < i\delta(H)$ *for every vertex* $y_j \in V(H)$.

Claim 10. $d_P(y_1) \geq 2$ *and* $d_P(y_h) \geq 2$.

Proof. If $d_P(y_1) = |\{x_r\}| = 1$, then $x_{r-1}, x_{r+1} \in N_G^2(y_1)$ and $d_H(y_1) = d_G(y_1) - 1$. Since $d_G(y_j) < i\delta(H)$ for every vertex $y_j \in V(H)$, we have $d_G(y_1) < id_G(y_1)$. Then, by the definition of $id_G(y_1)$, there is a vertex $y_s \in N_H(y_1)$ such that $d_G(y_s) \geq id_G(y_1) \geq i\delta(H)$, a contradiction.

If $d_P(y_h) = 1$, then, by a similar argument to the one above, we can obtain a contradiction. If $d_P(y_h) = 0$, then $d_H(y_h) \geq d_G(y_h)$. Since $d_G(y_j) < i\delta(H)$ for every vertex $y_j \in V(H)$, we have $d_G(y_h) < id_G(y_h)$. Then, by the definition of $id_G(y_h)$, there is a vertex $y_t \in N_H(y_h)$ such that $d_G(y_t) \geq id_G(y_h) \geq i\delta(H)$, a contradiction. □

Subcase 3. $|V(H)| \geq 2$.

Since P is a longest path of $\mathcal{T}^*$, we have $|N_P(y_1)^+| = |N_P(y_1)|$ and $N_P(y_1)^+ \subseteq N_G^2(y_1)$. Then, $|N_H(y_1) \cup N_P(y_1)^+| = d_G(y_1)$. Since $d_G(y_j) < i\delta(H)$ for every vertex $y_j \in V(H)$, $d_G(y_s) < id_G(y_1)$ for any $y_s \in N_H(y_1) \cup \{y_1\}$. By the definition of $id_G(y_1)$, there is a vertex $x_t \in N_P(y_1)^+$ such that $d_G(x_t) \geq id_G(y_1)$. Suppose x_i is the first vertex in $N_P(y_1)^+$ such that $d_G(x_i) \geq id_G(y_1)$ and x_j is the last vertex in $N_P(y_h)$. We can assume $i < j$. (Since, otherwise, there is a vertex $x_k \in N_P(y_h)^+$ such that $d_G(x_k) \geq id_G(y_h)$ and $k < i$.)

Let $Q_1 = x_2x_3\ldots x_{i-1}$, $Q_2 = x_{i+1}x_{i+2}\ldots x_{j-1}$, and $Q_3 = x_jx_{j+1}\ldots x_{p-1}$. Then, $n = |V(Q_1)| + |V(Q_2)| + |V(Q_3)| + |V(H)| + 3$

Claim 11. $\{x_1, x_p, x_i\}$ *is an independent set of* G.

Proof. If $x_1x_i \in E(G)$, then $y_h\overline{H}y_1x_{i-1}\overline{P}x_1x_iPx_p$ is a Hamiltonian path of G, a contradiction. If $x_1x_p \in E(G)$, then $y_1Hy_hx_jPx_px_1Px_{j-1}$ is a Hamiltonian path of G, a contradiction. If $x_ix_p \in E(G)$, then $x_1Px_{i-1}y_1Hy_hx_jPx_px_iPx_{j-1}$ is a Hamiltonian path of G, a contradiction. So, $\{x_1, x_p, x_i\}$ is an independent set of G. □

Claim 12. $d_{Q_1}(x_1) + d_{Q_1}(x_p) + d_{Q_1}(x_i) \leq |V(Q_1)| + 2$.

Proof. Set $A_2 = N_{Q_1}(x_1)^-, B_2 = N_{Q_1}(x_p)$ and $C_2 = N_{Q_1}(x_i)^+$.

If there is a vertex $x_s \in A_2 \cap C_2$, then $T' = T + \{x_jy_h, x_{i-1}y_1, x_1x_{s+1}, x_ix_{s-1}\} - \{x_sx_{s+1}, y_1x_r, x_ix_{i-1}, x_jx_{j-1}\}$ is a tree in $\mathcal{T}^*$ and $P' = x_p\overline{P}x_jy_h\overline{H}y_1x_{i-1}\overline{P}x_{s+1}x_1Px_{s-1}x_iPx_{j-1}$ is a path of $\mathcal{T}^*$ longer than P, this contradicts the choice of P. So, $A_2 \cap C_2 = \emptyset$.

If there is a vertex $x_s \in B_2 \cap C_2$, then $P' = x_1 P x_{s-1} x_i P x_j y_h \overline{H} y_1 x_{i-1} \overline{P} x_s x_p \overline{P} x_{j+1}$ is a Hamiltonian path of G, a contradiction. So, $B_2 \cap C_2 = \emptyset$. Notice that $A_2 \cap B_2 = \emptyset$ and $(N_{Q_1}(x_1) \setminus \{x_2\})^- \cup N_{Q_1}(x_p) \cup (N_{Q_1}(x_i) \setminus \{x_{i-1}\})^+ \subseteq V(Q_1)$. Therefore,

$$\begin{aligned}
& d_{Q_1}(x_1) + d_{Q_1}(x_p) + d_{Q_1}(x_i) \\
= \ & |N_{Q_1}(x_1)| + |N_{Q_1}(x_p)| + |N_{Q_1}(x_i)| \\
= \ & |(N_{Q_1}(x_1) \setminus \{x_2\})^-| + |\{x_2\}| + |N_{Q_1}(x_p)| + |(N_{Q_1}(x_i) \setminus \{x_{i-1}\})^+| + |\{x_{i-1}\}| \\
= \ & |(N_{Q_1}(x_1) \setminus \{x_2\})^- \cup N_{Q_1}(x_p) \cup (N_{Q_1}(x_i) \setminus \{x_{i-1}\})^+| + 2 \\
\leq \ & |V(Q_1)| + 2.
\end{aligned}$$

□

Claim 13. $d_{Q_2}(x_1) + d_{Q_2}(x_p) + d_{Q_2}(x_i) \leq |V(Q_2)| + 1$.

Proof. Since G has no Hamiltonian path, $x_p x_{j-1} \notin E(G)$. Set $A_3 = N_{Q_2}(x_i)^-, B_3 = N_{Q_2}(x_p)^+$, and $C_3 = N_{Q_2}(x_1)$.

If there is a vertex $x_s \in A_3 \cap B_3$, then $T' = T + \{x_p x_{s-1}, x_j y_h, x_{i-1} y_1, x_i x_{s+1}\} - \{x_s x_{s+1}, y_1 x_r, x_i x_{i-1}, x_j x_{j+1}\}$ is a tree in $\mathcal{T}^*$ and $P' = x_{j+1} P x_p x_{s-1} \overline{P} x_i x_{s+1} P x_j y_h \overline{H} y_1 x_{i-1} \overline{P} x_1$ is a path of $\mathcal{T}^*$ longer than P, this contradicts the choice of P. So, $A_3 \cap B_3 = \emptyset$. Similarly, $B_3 \cap C_3 = \emptyset$ and $A_3 \cap C_3 = \emptyset$. Note that $N_{Q_2}(x_1) \cup N_{Q_2}(x_p)^+ \cup (N_{Q_2}(x_i) \setminus \{x_{i+1}\})^- \subseteq V(Q_2)$. Therefore,

$$\begin{aligned}
& d_{Q_2}(x_1) + d_{Q_2}(x_p) + d_{Q_2}(x_i) \\
= \ & |N_{Q_2}(x_1)| + |N_{Q_2}(x_p)| + |N_{Q_2}(x_i)| \\
= \ & |N_{Q_2}(x_1)| + |N_{Q_2}(x_p)^+| + (|(N_{Q_2}(x_i) \setminus \{x_{i+1}\})^-| + |\{x_{i+1}\}| \\
= \ & |N_{Q_2}(x_1) \cup N_{Q_2}(x_p)^+ \cup (N_{Q_2}(x_i) \setminus \{x_{i+1}\})^-| + 1 \\
\leq \ & |V(Q_2)| + 1.
\end{aligned}$$

□

Claim 14. $d_{Q_3}(x_1) + d_{Q_3}(x_p) + d_{Q_3}(x_i) \leq |V(Q_3)| + 1$.

Proof. Since G has no Hamiltonian path, $x_1 x_{j-1}, x_1 x_{j+1} \notin E(G)$. By the choice of P, $y_h x_{j-1}, y_h x_{j+1} \notin E(G)$. Since G is $K_{1,4}$-free, $x_1 x_j \notin E(G)$ (otherwise $\{x_j, x_1, x_{j-1}, x_{j+1}, y_h\}$ induces a $K_{1,4}$). Set $A_4 = N_{Q_3}(x_1)^-, B_4 = N_{Q_3}(x_p)^+$, and $C_4 = N_{Q_3}(x_i)$.

If there is a vertex $x_s \in A_4 \cap B_4$, then $T' = T + \{x_p x_{s-1}, x_j y_h, x_1 x_{s+1}\} - \{x_s x_{s+1}, y_1 x_r, x_j x_{j-1}\}$ is a tree in $\mathcal{T}^*$ and $P' = x_{j-1} \overline{P} x_1 x_{s+1} P x_p x_{s-1} \overline{P} x_j y_h \overline{H} y_1$ is a path of $\mathcal{T}^*$ longer than P, this contradicts the choice of P. So, $A_4 \cap B_4 = \emptyset$. Similarly, $B_4 \cap C_4 = \emptyset$ and $A_4 \cap C_4 = \emptyset$. Note that $N_{Q_3}(x_1)^- \cup (N_{Q_3}(x_p) \setminus \{x_{p-1}\})^+ \cup N_{Q_3}(x_i) \subseteq V(Q_3)$. Therefore,

$$\begin{aligned}
& d_{Q_3}(x_1) + d_{Q_3}(x_p) + d_{Q_3}(x_i) \\
= \ & |N_{Q_3}(x_1)| + |N_{Q_3}(x_p)| + |N_{Q_3}(x_i)| \\
= \ & |N_{Q_3}(x_1)^-| + |(N_{Q_3}(x_p) \setminus \{x_{p-1}\})^+| + |\{x_{p-1}\}| + |N_{Q_3}(x_i)| \\
= \ & |N_{Q_3}(x_1)^- \cup (N_{Q_3}(x_p) \setminus \{x_{p-1}\})^+ \cup N_{Q_3}(x_i)| + 1 \\
\leq \ & |V(Q_3)| + 1.
\end{aligned}$$

□

Since P is a longest path of $\mathcal{T}^*$, it is easy to verify that $N_H(x_1) \cup N_H(x_p) \cup N_H(x_i) = \emptyset$. Therefore, by Claims 11–14, we have

$$\begin{aligned} d_G(x_1) + d_G(x_p) + d_G(x_i) &= \sum_{l=1}^{3}(d_{Q_l}(x_1) + d_{Q_l}(x_p) + d_{Q_l}(x_i)) \\ &\leq (|V(Q_1)| + 2) + (|V(Q_2)| + 1) + (|V(Q_3)| + 1) \\ &\leq n - 1. \end{aligned}$$

On the other hand, $d_G(x_1) + d_G(x_p) + d_G(x_i) \geq id_G(x_1) + id_G(x_p) + id_G(y_1) \geq i\sigma_3(G) \geq n$, a contradiction.

Subcase 4. $|V(H)| = 1$.

Then, $H = \{y_1\}$. By a similar argument to the one in Subcase 3, there is a vertex $x_t \in N_P(y_1)^+$ such that $d_G(x_t) \geq id_G(y_1)$. Suppose x_i is the first vertex in $N_P(y_1)^+$ such that $d_G(x_i) \geq id_G(y_1)$ and x_j is the last vertex in $N_P(y_1)$. Clearly, $i < j$. Similarly to the discussion in Claim 11, we can ascertain that $\{x_1, x_p, x_i\}$ is an independent set of G. Let $P_1 = x_1x_2 \ldots x_{i-1}$, $P_2 = x_{i+1}x_{i+2} \ldots x_{j-1}$, and $P_3 = x_jx_{j+1} \ldots x_{p-1}$.

If there is a vertex $x_s \in N_{P_1}(x_1)^- \cap N_{P_1}(x_i)$, then $P' = y_1x_{i-1}\overline{P}x_{s+1}x_1Px_sx_iPx_p$ is a Hamiltonian path of G, a contradiction. So, $N_{P_1}(x_1)^- \cap N_{P_1}(x_i) = \emptyset$. Similarly, $N_{P_1}(x_1)^- \cap N_{P_1}(x_p) = \emptyset$, $N_{P_1}(x_p)^- \cap N_{P_1}(x_i) = \emptyset$, and $N_{P_1}(x_i)^- \cap N_{P_1}(x_p) = \emptyset$. Since G is $K_{1,4}$-free, $N_{P_1}(x_i) \cap N_{P_1}(x_p) = \emptyset$ (otherwise, if there is a vertex $x_s \in N_{P_1}(x_i) \cap N_{P_1}(x_p)$, then $\{x_s, x_i, x_p, x_{s-1}, x_{s+1}\}$ induces a $K_{1,4}$, a contradiction). Therefore, by Lemma 4, $d_{P_1}(x_1) + d_{P_1}(x_p) + d_{P_1}(x_i) \leq |V(P_1)| + 1$. Similarly, $d_{P_2}(x_1) + d_{P_2}(x_p) + d_{P_2}(x_i) \leq |V(P_2)| + 1$ and $d_{P_3}(x_1) + d_{P_3}(x_p) + d_{P_3}(x_i) \leq |V(P_3)| + 1$. Thus, we have

$$\begin{aligned} d_G(x_1) + d_G(x_p) + d_G(x_i) &= \sum_{l=1}^{3}(d_{P_l}(x_1) + d_{P_l}(x_p) + d_{P_l}(x_i)) \\ &\leq \sum_{l=1}^{3}(|V(P_l)| + 1) = n - 1. \end{aligned}$$

On the other hand, $d_G(x_1) + d_G(x_p) + d_G(x_i) \geq id_G(x_1) + id_G(x_p) + id_G(y_1) \geq i\sigma_3(G) \geq n$, a contradiction. Now, the proof of Theorem 4 (1) is completed.

4. Discussion

Spanning k-ended trees are important in various fields such as network design, graph theory, and communication networks. They provide a structured way to connect all the nodes in a network while ensuring efficient communication and minimizing unnecessary connections. In addition, they serve as fundamental components for algorithms in routing, broadcasting, and spanning tree protocols. However, determining whether a connected graph has a spanning k-ended tree or not is NP-complete; therefore, it is important to identify sufficient conditions for the existence of such trees. The implicit-degree proposed by Zhu, Li, and Deng is an important indicator for the existence of spanning k-ended trees. In this article, we provide two sufficient conditions for $K_{1,4}$-free connected graphs to have spanning k-ended trees for $k = 2, 3$. Moreover, we point out that the lower bounds in our result are the best possible. There exist $K_{1,4}$-free connected graphs that satisfy our conditions but do not satisfy the conditions of Theorem 2; therefore, our result is stronger than that of Theorem 2.

5. Conclusions

From the definition of implicit-degree, it can be seen that the implicit-degree of a vertex comprehensively considers the degree of the vertex as well as the degrees of its neighbors and vertices at distance two with it. Furthermore, the implicit-degree of a vertex is greater than or equal to the degree of that vertex. In this article, we have demonstrated that the degree conditions in Theorem 2 can be equivalently replaced by implicit-degree conditions. Additionally, many classic results under degree conditions, such as Dirac's condition and Ore's condition for the Hamiltonian cycle problem, also hold when replaced by implicit-degree conditions. We believe that the existence problems of spanning subgraphs under degree conditions can be equivalently replaced by corresponding implicit-degree conditions.

Author Contributions: Conceptualization, J.C., C.-K.L., Q.S. and P.W.; Methodology, J.C., C.-K.L., Q.S. and P.W.; Writing—original draft preparation, J.C. and P.W.; Writing—Review and editing, J.C., C.-K.L., Q.S. and P.W.; Funding acquisition, J.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China of grant number 12371365.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bondy, J.A.; Murty, U.S.R. *Graph Theory, in Graduate Texts in Mathematics*; Springer: New York, NY, USA, 2008.
2. Ozeki, K.; Yamashita, T. Spanning Trees: A Survey. *Graphs Comb.* **2011**, *27*, 1–26. [CrossRef]
3. Broersma, H.; Tuinstra, H. Independent trees and Hamilton cycles. *J. Graph Theory* **1998**, *29*, 227–237. [CrossRef]
4. Chen, Y.; Chen, G.; Hu, Z. Spanning 3-ended trees in k-connected $K_{1,4}$-free graphs. *Sci. China Math.* **2014**, *57*, 1579–1586. [CrossRef]
5. Chen, Y.; Ha, P.; Hanh, D. Spanning trees with at most 4 leaves in $K_{1,5}$-free graphs. *Discret. Math.* **2019**, *342*, 2342–2349. [CrossRef]
6. Hu, Z.; Sun, P. Spanning 5-ended trees in $K_{1,5}$-free graphs. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 2565–2586. [CrossRef]
7. Tsugaki, M.; Yamashita, T. Spanning trees with few leaves. *Graphs Comb.* **2007**, *23*, 585–598. [CrossRef]
8. Wang, P.; Cai, J. Spanning trees with at most 4 leaves in $K_{1,5}$-free graphs. *Discret. Math.* **2019**, *342*, 1546–1552. [CrossRef]
9. Win, S. On a conjecture of Las Vergnas concerning certain spanning trees in graphs. *Results Math.* **1979**, *2*, 215–224. [CrossRef]
10. Matthews, M.; Sumner, D. Longest paths and cycles in $K_{1,3}$-free graphs. *J. Graph Theory* **1985**, *9*, 269–277. [CrossRef]
11. Kano, M.; Kyaw, A.; Matsude, H.; Ozeki, K.; Satio, A.; Yamashita, T. Spanning trees with a bounded number of leaves in claw-free graphs. *Ars Comb.* **2012**, *34*, 137–154.
12. Kyaw, A. Spanning trees with at most k leaves in $K_{1,4}$-free graphs. *Discret. Math.* **2011**, *311*, 2135–2142. [CrossRef]
13. Kyaw, A. Spanning trees with at most 3 leaves in $K_{1,4}$-free graphs. *Discret. Math.* **2009**, *309*, 6146–6148. [CrossRef]
14. Bondy, J.A. Largest cycles in graphs. *Discret. Math.* **1971**, *1*, 121–132. [CrossRef]
15. Zhu, Y.; Li, H.; Deng, X. Implicit-degrees and Circumferences. *Graphs Comb.* **1989**, *5*, 283–290.
16. Huang, X. Implicit-degree condition for hamiltonicity of 2-heavy graphs. *Discret. Appl. Math.* **2017**, *219*, 126–131. [CrossRef]
17. Li, H. Generalizations of Dirac's theorem in Hamiltonian graph theory-A survey. *Discret. Math.* **2013**, *313*, 2034–2053. [CrossRef]
18. Cai, J.; Flandrin, E.; Li, H.; Sun, Q. Spanning trees with a bounded number of leaves. *Opusc. Math.* **2017**, *37*, 501–508. [CrossRef]