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Killing and 2-Killing Vector Fields on Doubly Warped Products

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Abstract: We provide a condition for a 2-Killing vector field on a compact Riemannian manifold to be Killing and apply the result to doubly warped product manifolds. We establish a connection between the property of a vector field on a doubly warped product manifold and its components on the factor manifolds to be Killing or 2-Killing. We also prove that a Killing vector field on the doubly warped product gives rise to a Ricci soliton factor manifold if and only if it is an Einstein manifold. If a component of a Killing vector field on the doubly warped product is of a gradient type, then, under certain conditions, the corresponding factor manifold is isometric to the Euclidean space. Moreover, we provide necessary and sufficient conditions for a doubly warped product to reduce to a direct product. As applications, we characterize the 2-Killing vector fields on the doubly warped spacetimes, particularly on the standard static spacetime and on the generalized Robertson–Walker spacetime.

Keywords: 2-Killing vector field; doubly warped product manifold; spacetime; Ricci soliton

MSC: 35Q51; 53B50; 83C05; 83E15

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1. Introduction

Killing vector fields are infinitesimal generators of isometries on a Riemannian manifold (M, g). They satisfy the condition $\pounds_\xi g = 0$, i.e., the Lie derivative of the metric in their direction to vanish. A more general notion, namely, the 2-Killing vector field, has been introduced in [1] as being a smooth vector field ξ such that $\pounds_\xi \pounds_\xi g = 0$. Obviously, any Killing vector field is also 2-Killing, but in general, the converse is not true. Therefore, a first question which can arise is the following: *When does a 2-Killing vector field reduces to a Killing vector field?* In his paper [1], Oprea established some relations between them, and he gave a necessary and sufficient condition in terms of the Ricci curvature of the Levi-Civita connection of g for a 2-Killing vector field to be parallel, hence, a Killing vector field.

On the other hand, solitary waves preserving their shape while propagating with a constant speed, which are known as solitons, are mathematically modeled as stationary solutions to a certain flow. They are described by a second-order nonlinear equation and have concrete applications in nonlinear dynamics. In Riemannian geometry, Ricci solitons correspond to the Ricci flow, which was first considered by Hamilton in [2]. The Ricci soliton equation is

$$\frac{1}{2}\pounds_\xi g + \mathrm{Ric} = \lambda g$$

with λ as a real constant, which reduces to the Einstein condition when ξ is a Killing vector field. Another type of recently considered soliton, namely, the hyperbolic Ricci soliton [3], which is defined by

$$\pounds_\xi \pounds_\xi g + \mu \pounds_\xi g + \mathrm{Ric} = \lambda g$$

with λ and μ two real constants, also reduces to an Einstein manifold if ξ is a Killing vector field and to a Ricci soliton if ξ is a 2-Killing vector field.

Spacetimes, which are mathematically described as Lorentzian manifolds, model the physical world. Through the important particular spacetimes counts those that occur when the manifold is a warped product of a certain type: either a generalized Robertson–Walker spacetime $I \times_f M$ or a standard static spacetime $M \times_f I$ [4]. It is worth mentioning that the manifolds of warped product type, introduced by Bishop and O'Neill [5], are of special interest in physics. Doubly warped products are some of their generalizations [6,7].

It is known that the existence of distinguished vector fields on Riemannian manifolds dictates their geometry. Killing vector fields on doubly warped products have already been treated in [6,8]. In the present paper, which extends some of the above-mentioned results, we firstly provide a necessary and sufficient condition for a vector field, which satisfies the condition that the second Lie derivative of the metric in its direction is traceless (in particular, a 2-Killing vector field) to be a parallel vector field, hence a Killing vector field. We consider Killing and 2-Killing vector fields on doubly warped products, and we establish some consequences on the factor manifolds. In particular, we characterize the factor manifolds as being isometric to the Euclidean space when there exists a Killing vector field on the doubly warped product, and we provide the necessary and sufficient conditions for a doubly warped product to reduce to a direct product when there exists a 2-Killing vector field on the factor manifolds. We also show that, under a certain assumption, a factor manifold is a Ricci soliton having as a potential vector field the component of a Killing vector field on the doubly warped product manifold if and only if it is Einstein, and we show that an Einstein factor (M_i, g_i), $i = 1, 2$ of a doubly warped product endowed with a 2-Killing vector field $\xi = \xi_1 + \xi_2$ is a nontrivial hyperbolic Ricci soliton with the potential vector field ξ_i, $i = 1, 2$. As physical applications, we characterize Killing and 2-Killing vector fields on doubly warped spacetimes and, in particular, on the standard static spacetime and on the generalized Robertson–Walker spacetime. Our paper continues and completes with new properties that expand on the results obtained in [8,9], where the authors also provided some characterizations of the Killing and 2-Killing vector fields on the standard static spacetime and on the generalized Robertson–Walker spacetime.

2. On 2-Killing Vector Fields

A smooth vector field ξ on a pseudo-Riemannian manifold (M, g) is called Killing if $\pounds_\xi g = 0$, and it is called 2-Killing [1] if

$$\pounds_\xi \pounds_\xi g = 0,$$

where $\pounds_\xi T$ stands for the Lie derivative of a symmetric $(0,2)$-tensor field T in the direction of ξ, which is given by

$$(\pounds_\xi T)(X, Y) := \xi(T(X, Y)) - T([\xi, X], Y) - T(X, [\xi, Y])$$

for any smooth vector fields X, Y on M.

Any Killing vector field is 2-Killing too, but not conversely. Nontrivial cases when a 2-Killing vector field is also Killing were given in [1,10]. We shall extend here some of those results.

Lemma 1. *Let ξ, X, Y be smooth vector fields on (M, g). Then,*

$$\begin{aligned}(\pounds_\xi \pounds_\xi g)(X, Y) = {} & g(\nabla^2_{\xi,X}\xi, Y) + g(X, \nabla^2_{\xi,Y}\xi) + 2g(\nabla_X \xi, \nabla_Y \xi) \\ & + g(\nabla_{\nabla_X \xi}\xi, Y) + g(X, \nabla_{\nabla_Y \xi}\xi),\end{aligned}$$

where $\nabla^2_{X,Y}\xi := \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi$, and ∇ stands for the Levi-Civita connection of g.

Proof. We have

$$\begin{aligned}(\pounds_\xi \pounds_\xi g)(X,Y) &= \xi((\pounds_\xi g)(X,Y)) - (\pounds_\xi g)([\xi,X],Y) - (\pounds_\xi g)(X,[\xi,Y]) \\ &= \xi\big(g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi)\big) - \big(g(\nabla_{[\xi,X]}\xi,Y) + g([\xi,X],\nabla_Y\xi)\big) \\ &\quad - \big(g(\nabla_X\xi,[\xi,Y]) + g(X,\nabla_{[\xi,Y]}\xi)\big) \\ &= g(\nabla_\xi\nabla_X\xi,Y) + g(X,\nabla_\xi\nabla_Y\xi) - g(\nabla_{\nabla_\xi X}\xi,Y) + g(\nabla_{\nabla_X\xi}\xi,Y) \\ &\quad + 2g(\nabla_X\xi,\nabla_Y\xi) - g(X,\nabla_{\nabla_\xi Y}\xi) + g(X,\nabla_{\nabla_Y\xi}\xi). \quad \square\end{aligned}$$

In particular, for $X = Y$, taking into account that the Riemannian curvature R of ∇ satisfies

$$R(\xi,X)\xi = \nabla^2_{\xi,X}\xi - \nabla^2_{X,\xi}\xi,$$

we obtain the conclusion.

Lemma 2 ([1])**.** *Let ξ, X be smooth vector fields on (M,g). Then,*

$$(\pounds_\xi \pounds_\xi g)(X,X) = 2\big[g(R(\xi,X)\xi,X) + g(\nabla_X\nabla_\xi\xi,X) + g(\nabla_X\xi,\nabla_X\xi)\big].$$

The result of Theorem 2.2 from [1] can be extended as follows.

Proposition 1. *Let ξ be a smooth vector field on a compact Riemannian manifold (M,g). If* $\mathrm{trace}(\pounds_\xi \pounds_\xi g) = 0$ *and* $\int_M \mathrm{Ric}(\xi,\xi) \leq 0$*, then ξ is a parallel vector field.*

Proof. Let $\{e_i\}_{1\leq i\leq n}$ be a local orthonormal frame field on M, where $n = \dim(M)$. Then, we have

$$(\pounds_\xi \pounds_\xi g)(e_i,e_i) = 2\big[-g(R(e_i,\xi)\xi,e_i) + g(\nabla_{e_i}\nabla_\xi\xi,e_i) + g(\nabla_{e_i}\xi,\nabla_{e_i}\xi)\big]$$

from Lemma 2. By taking the sum in the previous relation, we obtain

$$-\mathrm{Ric}(\xi,\xi) + \mathrm{div}(\nabla_\xi\xi) + \|\nabla\xi\|^2 = \frac{1}{2}\,\mathrm{trace}(\pounds_\xi \pounds_\xi g)(=0).$$

By integrating the relation above with respect to the canonical measure, we find

$$\int_M \|\nabla\xi\|^2 = \int_M \mathrm{Ric}(\xi,\xi) \leq 0,$$

and, hence, $\nabla\xi = 0$. $\square$

From the previous proposition, we can state the following theorem.

Theorem 1. *If ξ is a smooth vector field on a compact Riemannian manifold (M,g) such that* $\mathrm{trace}(\pounds_\xi \pounds_\xi g) = 0$*, then ξ is a parallel vector field if and only if* $\int_M \mathrm{Ric}(\xi,\xi) = 0$*.*

3. Describing 2-Killing Vector Fields on Doubly Warped Products

Let (M_1,g_1) and (M_2,g_2) be two pseudo-Riemannian manifolds, let ∇^1 and ∇^2 be the Levi-Civita connections of g_1 and g_2, respectively, and let f_1 and f_2 be two positive smooth functions on M_1 and M_2, respectively. We consider the *doubly warped product manifold* ${}_{f_2}M_1 \times_{f_1} M_2 =: (\tilde{M},\tilde{g})$, which is defined [6,7] as

$$\tilde{M} := M_1 \times M_2, \;\; \tilde{g} := (\pi_2^*(f_2))^2\pi_1^*(g_1) + (\pi_1^*(f_1))^2\pi_2^*(g_2)$$

and where $\pi_i : M_1 \times M_2 \to M_i$ is the canonical projection, $i = 1,2$. If only one of f_1 and f_2 is a constant, then $(\tilde{M},\tilde{g})$ is a warped product manifold (see [5]). Moreover, if both f_1 and f_2 are constants, then $(\tilde{M},\tilde{g})$ is a direct product manifold (and we call it the trivial case).

For the rest of this paper, we shall use the same notation for a function on M_i, $i=1,2$ and its pullback on $\tilde{M}$, for a metric on M_i, $i=1,2$ and its pullback on $\tilde{M}$, and also for a vector field on M_i, $i=1,2$ and its lift on $\tilde{M}$. The set of smooth sections of a smooth manifold M will be denoted by $\Gamma(TM)$.

We have the orthogonal decomposition

$$T\tilde{M} = TM_1 \oplus TM_2\,,$$

and for any $\tilde{X} \in \Gamma(T\tilde{M})$, we denote

$$\tilde{X} = X_1 + X_2,$$

where $X_i \in \Gamma(TM_i)$, $i=1,2$.

The expressions of the first and the second Lie derivatives are given in the following proposition.

Proposition 2. *Let $\xi = \xi_1 + \xi_2 \in \Gamma(T\tilde{M})$. Then, for any $X = X_1 + X_2$, $Y = Y_1 + Y_2 \in \Gamma(T\tilde{M})$, we have:*

$$\begin{aligned}
(\pounds_\xi \tilde{g})(X,Y) &= f_2^2(\pounds_{\xi_1} g_1)(X_1,Y_1) + f_1^2(\pounds_{\xi_2} g_2)(X_2,Y_2)\\
&\quad + \xi_2(f_2^2) g_1(X_1,Y_1) + \xi_1(f_1^2) g_2(X_2,Y_2),\\
(\pounds_\xi \pounds_\xi \tilde{g})(X,Y) &= f_2^2(\pounds_{\xi_1}\pounds_{\xi_1} g_1)(X_1,Y_1) + f_1^2(\pounds_{\xi_2}\pounds_{\xi_2} g_2)(X_2,Y_2)\\
&\quad + 2\xi_2(f_2^2)(\pounds_{\xi_1} g_1)(X_1,Y_1) + 2\xi_1(f_1^2)(\pounds_{\xi_2} g_2)(X_2,Y_2)\\
&\quad + \xi_2(\xi_2(f_2^2)) g_1(X_1,Y_1) + \xi_1(\xi_1(f_1^2)) g_2(X_2,Y_2).
\end{aligned}$$

Proof. The expression of the first Lie derivative on a doubly warped product manifold has been previously given in [8].

For $\xi = \xi_1 + \xi_2$, $X = X_1 + X_2$, $Y = Y_1 + Y_2 \in \Gamma(T\tilde{M})$, we have

$$[\xi, X] = [\xi_1, X_1] + [\xi_2, X_2].$$

Then,

$$\begin{aligned}
(\pounds_\xi \pounds_\xi \tilde{g})(X,Y) :&= \xi((\pounds_\xi \tilde{g})(X,Y)) - (\pounds_\xi \tilde{g})([\xi,X],Y) - (\pounds_\xi \tilde{g})(X,[\xi,Y])\\
&= \xi_2(\xi_2(f_2^2)) g_1(X_1,Y_1) + \xi_2(f_2^2)\xi_1(g_1(X_1,Y_1))\\
&\quad + \xi_2(f_2^2)(\pounds_{\xi_1} g_1)(X_1,Y_1) + f_2^2\xi_1\big((\pounds_{\xi_1} g_1)(X_1,Y_1)\big)\\
&\quad + \xi_1(\xi_1(f_1^2)) g_2(X_2,Y_2) + \xi_1(f_1^2)\xi_2(g_2(X_2,Y_2))\\
&\quad + \xi_1(f_1^2)(\pounds_{\xi_2} g_2)(X_2,Y_2) + f_1^2\xi_2\big((\pounds_{\xi_2} g_2)(X_2,Y_2)\big)\\
&\quad - \xi_2(f_2^2) g_1([\xi_1,X_1],Y_1) - f_2^2(\pounds_{\xi_1} g_1)([\xi_1,X_1],Y_1)\\
&\quad - \xi_1(f_1^2) g_2([\xi_2,X_2],Y_2) - f_1^2(\pounds_{\xi_2} g_2)([\xi_2,X_2],Y_2)\\
&\quad - \xi_2(f_2^2) g_1(X_1,[\xi_1,Y_1]) - f_2^2(\pounds_{\xi_1} g_1)(X_1,[\xi_1,Y_1])\\
&\quad - \xi_1(f_1^2) g_2(X_2,[\xi_2,Y_2]) - f_1^2(\pounds_{\xi_2} g_2)(X_2,[\xi_2,Y_2])\\
&= \xi_2(\xi_2(f_2^2)) g_1(X_1,Y_1) + 2\xi_2(f_2^2)(\pounds_{\xi_1} g_1)(X_1,Y_1)\\
&\quad + f_2^2(\pounds_{\xi_1}\pounds_{\xi_1} g_1)(X_1,Y_1) + \xi_1(\xi_1(f_1^2)) g_2(X_2,Y_2)\\
&\quad + 2\xi_1(f_1^2)(\pounds_{\xi_2} g_2)(X_2,Y_2) + f_1^2(\pounds_{\xi_2}\pounds_{\xi_2} g_2)(X_2,Y_2),
\end{aligned}$$

and the proof is complete. □

If one of the components of ξ is Killing or 2-Killing, or if ξ is a Killing or a 2-Killing vector field, then from the above proposition, we deduce the following proposition.

Proposition 3.

(i) *If ξ_1 is a Killing vector field on (M_1, g_1), then we have the following:*
   (a) *its lift is a Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$ if and only if $\xi_1(f_1) = 0$;*
   (b) *its lift is a 2-Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$ if and only if*
$$f_1\xi_1(\xi_1(f_1)) + (\xi_1(f_1))^2 = 0.$$
(ii) *If ξ_1 and ξ_2 are Killing vector fields on (M_1, g_1) and (M_2, g_2), respectively, then we have the following:*
   (a) *$\xi = \xi_1 + \xi_2$ is a Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$ if and only if $\xi_1(f_1) = 0$ and $\xi_2(f_2) = 0$;*
   (b) *$\xi = \xi_1 + \xi_2$ is a 2-Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$ if and only if*
$$f_1\xi_1(\xi_1(f_1)) + (\xi_1(f_1))^2 = 0 \ \textit{and} \ f_2\xi_2(\xi_2(f_2)) + (\xi_2(f_2))^2 = 0.$$
(iii) *If ξ_1 is a 2-Killing vector field on (M_1, g_1), then its lift is a 2-Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$ if and only if*
$$f_1\xi_1(\xi_1(f_1)) + (\xi_1(f_1))^2 = 0.$$
(iv) *If ξ_1 and ξ_2 are 2-Killing vector fields on (M_1, g_1) and (M_2, g_2), respectively, then $\xi = \xi_1 + \xi_2$ is a 2-Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$ if and only if*
$$2\xi_2(f_2^2)\pounds_{\xi_1}g_1 = -\xi_2(\xi_2(f_2^2))g_1 \ \textit{and} \ 2\xi_1(f_1^2)\pounds_{\xi_2}g_2 = -\xi_1(\xi_1(f_1^2))g_2.$$

As a consequence, we have the following corollary.

Corollary 1 ([9]). *If $M_1 \times_{f_1} M_2$ is a warped product manifold, then we have the following:*

(i) *ξ_2 is a Killing vector field on (M_2, g_2) if and only if its lift is a Killing vector field on $M_1 \times_{f_1} M_2$;*
(ii) *ξ_2 is a 2-Killing vector field on (M_2, g_2) if and only if its lift is a 2-Killing vector field on $M_1 \times_{f_1} M_2$.*

Proposition 4.

(i) *If $\xi = \xi_1 + \xi_2$ is a Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$, then ξ_1 is a Killing vector field on (M_1, g_1) if and only if $\xi_2(f_2) = 0$.*
(ii) *If $\xi = \xi_1 + \xi_2$ is a 2-Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$, then ξ_1 is a 2-Killing vector field on (M_1, g_1) if and only if*
$$2\xi_2(f_2^2)\pounds_{\xi_1}g_1 = -\xi_2(\xi_2(f_2^2))g_1.$$

As a consequence, we have the following corollary.

Corollary 2 ([9]).

(i) *If $\xi = \xi_1 + \xi_2$ is a Killing vector field on the warped product $M_1 \times_{f_1} M_2$, then ξ_1 is a Killing vector field on (M_1, g_1).*
(ii) *If $\xi = \xi_1 + \xi_2$ is a 2-Killing vector field on the warped product $M_1 \times_{f_1} M_2$, then ξ_1 is a 2-Killing vector field on (M_1, g_1).*

Example 1. *We consider the Heisenberg group $M := \{(x, y, z) \in \mathbb{R}^3, \ z > 0\}$ equipped with the Riemannian metric $g := dx \otimes dx + dy \otimes dy + \eta \otimes \eta$, where (x, y, z) denote the standard coordinates in $\mathbb{R}^3$, and $\eta := ydx - xdy + dz$. Then, (ϕ, ξ, η, g) defines a Sasakian structure on M, where*
$$\phi := \left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right) \otimes dx - \left(\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}\right) \otimes dy, \ \ \xi := \frac{\partial}{\partial z},$$

and ξ is a Killing, hence, a 2-Killing vector field on M.

Let ${}_{f_2}I \times_{f_1} M$ be the doubly warped spacetime, where I is a connected open real interval equipped with the metric $-dt^2$, and $f_2 : M \to \mathbb{R}$, $f_2(x, y, z) := \sqrt{c_1 z + c_2}$, with $c_1, c_2 \in \mathbb{R}^*_+$.

Then, according to Proposition 3 (i)(b), the lift of ξ to ${}_{f_2}I \times_{f_1} M$ is a 2-Killing vector field. On the other hand, if we take f_2 as a constant, then the lift of ξ to the standard static spacetime is a Killing, hence, a 2-Killing vector field.

We shall further characterize the factor manifolds of a doubly warped product as being Ricci-type solitons having as a potential vector field the component of a Killing or a 2-Killing vector field on the doubly warped product.

We recall that (M, g, ξ, λ) is called a *Ricci soliton* [2] if the vector field ξ and the scalar $\lambda \in \mathbb{R}$ satisfy

$$\frac{1}{2}\pounds_\xi g + \mathrm{Ric} = \lambda g,$$

where Ric is the Ricci curvature of (M, g).

Theorem 2. *Let $\xi = \xi_1 + \xi_2$ be a Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$. Then, we have the following:*

(i) *$(M_1, g_1, \xi_1, \lambda_1)$ is a Ricci soliton if and only if (M_1, g_1) is an Einstein manifold; in this case, $\lambda_1 = \frac{r_1}{n_1} - \xi_2(\ln f_2)$, where r_1 is the scalar curvature of (M_1, g_1) and $n_1 = \dim(M_1) > 2$;*

(ii) *if (M_1, g_1) is a complete Riemannian manifold, $\xi_1 = \nabla^1 h_1$ for h_1 a smooth function on M_1, and $\xi_2(\ln f_2)$ is a nonzero constant, then, (M_1, g_1) is isometric to the Euclidean space.*

Proof. If $(M_1, g_1, \xi_1, \lambda_1)$ is a Ricci soliton, then, for any $X_1, Y_1 \in \Gamma(TM_1)$, we have

$$\frac{1}{2}(\pounds_{\xi_1} g_1)(X_1, Y_1) + \mathrm{Ric}_1(X_1, Y_1) = \lambda_1 g_1(X_1, Y_1),$$

and, from Proposition 2, we obtain

$$\mathrm{Ric}_1(X_1, Y_1) = \left(\lambda_1 + \frac{\xi_2(f_2^2)}{2f_2^2}\right) g_1(X_1, Y_1),$$

which, by taking the trace, gives us

$$r_1 = n_1\left(\lambda_1 + \frac{\xi_2(f_2)}{f_2}\right).$$

Conversely, if $\mathrm{Ric}_1 = \frac{r_1}{n_1} g_1$, then

$$\begin{aligned}\frac{1}{2}(\pounds_{\xi_1} g_1)(X_1, Y_1) + \mathrm{Ric}_1(X_1, Y_1) &= -\frac{\xi_2(f_2^2)}{2f_2^2} g_1(X_1, Y_1) + \frac{r_1}{n_1} g_1(X_1, Y_1) \\ &= \lambda_1 g_1(X_1, Y_1)\end{aligned}$$

with $\lambda_1 = \frac{r_1}{n_1} - \frac{\xi_2(f_2^2)}{2f_2^2}$, and we obtain (i).

For (ii), we notice that

$$\mathrm{Hess}(h_1)(X_1, Y_1) = -\xi_2(\ln f_2) g_1(X_1, Y_1)$$

for any $X_1, Y_1 \in \Gamma(TM_1)$, and we obtain the conclusion from Tashiro's theorem [11]. □

We recall that $(M, g, \xi, \lambda, \mu)$ is called a *hyperbolic Ricci soliton* [3] if the vector field ξ and the scalars $\lambda, \mu \in \mathbb{R}$ satisfy

$$£_\xi £_\xi g + \mu £_\xi g + \mathrm{Ric} = \lambda g,$$

where Ric is the Ricci curvature of (M, g).

We remark that if ξ is a Killing vector field, then the manifold is an Einstein manifold provided that $\dim(M) > 2$, and if ξ is a 2-Killing vector field, then the manifold is a Ricci soliton. If none of these particular cases occurs, we shall call the soliton nontrivial.

Theorem 3. *Let $\xi = \xi_1 + \xi_2$ be a 2-Killing vector field on ${}_{f_2}M_1 \times_{f_1} M_2$. If $\xi_2(\ln f_2) =: c$ is a nonzero constant and (M_1, g_1) is an Einstein manifold, then $(M_1, g_1, \xi_1, \lambda_1, \mu_1)$ is a nontrivial hyperbolic Ricci soliton; in this case, $\lambda_1 = \frac{r_1}{n_1} - 4c^2$ and $\mu_1 = 4c$, where r_1 is the scalar curvature of (M_1, g_1), and $n_1 = \dim(M_1)$.*

Proof. From Proposition 2, we obtain that

$$(£_{\xi_1} £_{\xi_1} g_1)(X_1, Y_1) + 2\frac{\xi_2(f_2^2)}{f_2^2}(£_{\xi_1} g_1)(X_1, Y_1) = -\frac{\xi_2(\xi_2(f_2^2))}{f_2^2} g_1(X_1, Y_1)$$

for any $X_1, Y_1 \in \Gamma(TM_1)$. Since

$$\frac{\xi_2(f_2^2)}{f_2^2} = 2\frac{\xi_2(f_2)}{f_2} = 2\xi_2(\ln f_2) = 2c,$$
$$\frac{\xi_2(\xi_2(f_2^2))}{f_2^2} = 2\Big[\Big(\frac{\xi_2(f_2)}{f_2}\Big)^2 + \frac{\xi_2(\xi_2(f_2))}{f_2}\Big] = 4c^2,$$

and $\mathrm{Ric}_1 = \frac{r_1}{n_1} g_1$, we obtain

$$(£_{\xi_1} £_{\xi_1} g_1)(X_1, Y_1) + 4c(£_{\xi_1} g_1)(X_1, Y_1) + \mathrm{Ric}_1(X_1, Y_1) = \Big(\frac{r_1}{n_1} - 4c^2\Big) g_1(X_1, Y_1)$$

for any $X_1, Y_1 \in \Gamma(TM_1)$; hence, we have the conclusion. □

Also, we have the following corollary.

Corollary 3. *If $\xi = \xi_1 + \xi_2$ is a 2-Killing vector field on the warped product $M_1 \times_{f_1} M_2$ and (M_1, g_1) is an Einstein manifold, then $(M_1, g_1, \xi_1, \lambda_1, \mu_1)$ is a nontrivial hyperbolic Ricci soliton; in this case, $\lambda_1 = \frac{r_1}{n_1}$, and $\mu_1 = 0$, where r_1 is the scalar curvature of (M_1, g_1), and $n_1 = \dim(M_1)$.*

Now, we will provide necessary and sufficient conditions for a doubly warped product manifold to reduce to a direct product.

Theorem 4. *Let ξ_i be a smooth vector field on the connected Riemannian factor manifold (M_i, g_i), $i = 1, 2$, of the doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$, such that $\xi_i(\ln f_i)$=0 for $i = 1, 2$. Then, $\xi = \xi_1 + \xi_2$ is a parallel vector field if and only if ξ_i is a parallel vector field on (M_i, g_i) and f_i is a constant for $i = 1, 2$, i.e., the manifold is a direct product manifold.*

Proof. Let $\{e_1, \ldots, e_{n_1}\}$ and $\{e_{n_1+1}, \ldots, e_{n_1+n_2}\}$ be local orthonormal frame fields on (M_1, g_1) and (M_2, g_2), respectively, where $n_i = \dim(M_i)$ for $i = 1, 2$. Then,

$$\Big\{\tilde{e}_1 := \frac{1}{f_2} e_1, \ldots, \tilde{e}_{n_1} := \frac{1}{f_2} e_{n_1}, \tilde{e}_{n_1+1} := \frac{1}{f_1} e_{n_1+1}, \ldots, \tilde{e}_{n_1+n_2} := \frac{1}{f_1} e_{n_1+n_2}\Big\}$$

is a local orthonormal frame field on ${}_{f_2}M_1 \times_{f_1} M_2$. We have

$$
\begin{aligned}
\|\tilde{\nabla}\xi\|_{\tilde{g}}^2 :&= \sum_{i=1}^{n_1+n_2} \tilde{g}(\tilde{\nabla}_{\tilde{e}_i}\xi, \tilde{\nabla}_{\tilde{e}_i}\xi) \\
&= \sum_{i=1}^{n_1} \tilde{g}(\tilde{\nabla}_{\tilde{e}_i}\xi, \tilde{\nabla}_{\tilde{e}_i}\xi) + \sum_{i=n_1+1}^{n_1+n_2} \tilde{g}(\tilde{\nabla}_{\tilde{e}_i}\xi, \tilde{\nabla}_{\tilde{e}_i}\xi) \\
&= \frac{1}{f_2^2} \sum_{i=1}^{n_1} \Big[\tilde{g}(\tilde{\nabla}_{e_i}\xi_1, \tilde{\nabla}_{e_i}\xi_1) + \tilde{g}(\tilde{\nabla}_{e_i}\xi_2, \tilde{\nabla}_{e_i}\xi_2) + 2\tilde{g}(\tilde{\nabla}_{e_i}\xi_1, \tilde{\nabla}_{e_i}\xi_2)\Big] \\
&\quad + \frac{1}{f_1^2} \sum_{i=n_1+1}^{n_1+n_2} \Big[\tilde{g}(\tilde{\nabla}_{e_i}\xi_1, \tilde{\nabla}_{e_i}\xi_1) + \tilde{g}(\tilde{\nabla}_{e_i}\xi_2, \tilde{\nabla}_{e_i}\xi_2) + 2\tilde{g}(\tilde{\nabla}_{e_i}\xi_1, \tilde{\nabla}_{e_i}\xi_2)\Big].
\end{aligned}
$$

For $j \neq i$, we have [6]

$$
\begin{cases}
\tilde{\nabla}_X Y = \nabla^i_X Y - \tilde{g}(X,Y)\nabla(\ln f_j), \ (\forall)\ X, Y \in \Gamma(TM_i), \\
\tilde{\nabla}_X Y = X(\ln f_i)Y + Y(\ln f_j)X, \ (\forall)\ X \in \Gamma(TM_i), Y \in \Gamma(TM_j),
\end{cases}
$$

where ∇f denotes the gradient of a function f on the doubly warped product manifold.

After a long computation, we obtain

$$
\begin{aligned}
\|\tilde{\nabla}\xi\|_{\tilde{g}}^2 = \|\nabla^1\xi_1\|_1^2 &+ f_2^2\|\xi_1\|_1^2 \cdot \|\nabla(\ln f_2)\|_{\tilde{g}}^2 + f_1^2\|\xi_2\|_2^2 \cdot \|\nabla(\ln f_1)\|_{\tilde{g}}^2 \\
&+ n_1(\xi_2(\ln f_2))^2 + 2\xi_2(\ln f_2)\Big(\mathrm{div}^1(\xi_1) - \xi_1(\ln f_1)\Big) \\
&+ f_2^2\|\xi_1\|_1^2 \cdot \|\nabla(\ln f_2)\|_{\tilde{g}}^2 + n_2(\xi_1(\ln f_1))^2 + \|\nabla^2\xi_2\|_2^2 \\
&+ f_1^2\|\xi_2\|_2^2 \cdot \|\nabla(\ln f_1)\|_{\tilde{g}}^2 + 2\xi_1(\ln f_1)\Big(\mathrm{div}^2(\xi_2) - \xi_2(\ln f_2)\Big).
\end{aligned}
$$

Considering the hypotheses, we infer

$$
\begin{aligned}
\|\tilde{\nabla}\xi\|_{\tilde{g}}^2 = \|\nabla^1\xi_1\|_1^2 &+ \|\nabla^2\xi_2\|_2^2 \\
&+ 2\Big[f_2^2\|\xi_1\|_1^2 \cdot \|\nabla(\ln f_2)\|_{\tilde{g}}^2 + f_1^2\|\xi_2\|_2^2 \cdot \|\nabla(\ln f_1)\|_{\tilde{g}}^2\Big],
\end{aligned}
$$

and we obtain the conclusion. □

As a consequence, we have the following corollary.

Corollary 4. *There do not exist nontrivial doubly warped products ${}_{f_2}M_1 \times_{f_1} M_2$ with connected Riemannian factor manifolds admitting a parallel vector field $\xi = \xi_1 + \xi_2$ satisfying $\nabla^i\xi_i = 0$ and $\xi_i(\ln f_i) = 0$ for $i = 1, 2$.*

Now, using Proposition 1 and Theorem 4, we extend from warped products to doubly warped products (and use a weaker hypothesis) Shenawy and Ünal's result from Theorem 3.2 of [9].

Theorem 5. *Let ξ_i be a smooth vector field on the compact and connected Riemannian factor manifold (M_i, g_i), $i = 1, 2$, of the doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ such that $\xi_i(\ln f_i) = 0$ for $i = 1, 2$. If $\mathrm{trace}(\pounds_{\xi_i}\pounds_{\xi_i} g_i) = 0$ and $\int_M \mathrm{Ric}(\xi_i, \xi_i) \le 0$ for $i = 1, 2$, then $\xi = \xi_1 + \xi_2$ is a parallel vector field if and only if f_i is a constant for $i = 1, 2$, i.e., the manifold is a direct product manifold.*

Proof. From Proposition 1, we deduce that ξ_i is a parallel vector field on (M_i, g_i), $i = 1, 2$, and, considering the hypotheses, we infer

$$
\|\tilde{\nabla}\xi\|_{\tilde{g}}^2 = 2\Big[f_2^2\|\xi_1\|_1^2 \cdot \|\nabla(\ln f_2)\|_{\tilde{g}}^2 + f_1^2\|\xi_2\|_2^2 \cdot \|\nabla(\ln f_1)\|_{\tilde{g}}^2\Big]
$$

from Theorem 4; hence, we have the conclusion. □

Remark 1. *Since the behavior with respect to the two factor manifolds of a doubly warped product is the same, all the results obtained in this section regarding one of the factors are also valid for the other one.*

4. The Spacetime Case

Let ${}_{f_2}I \times_{f_1} M$ be a doubly warped spacetime, where I is a connected open real interval equipped with the metric $g_0 = -dt^2$, and (M, g) is a three-dimensional Riemannian manifold. If f_1 is a constant, then the manifold is the standard static spacetime, and if f_2 is a constant, then it is the generalized Robertson–Walker spacetime.

We will consider both the Killing and 2-Killing cases, wherein some properties for the Killing case being studied are also found in [7,8] and for the 2-Killing case are found in [9].

Lemma 3. *If $\xi = u\frac{d}{dt}$, with u being a smooth function on I, then*

$$\pounds_\xi g_0 = 2\frac{du}{dt} g_0;$$
$$\pounds_\xi \pounds_\xi g_0 = 2\left[u\frac{d^2u}{dt^2} + 2\left(\frac{du}{dt}\right)^2\right] g_0.$$

Proof. For any vector fields $X = v\frac{d}{dt}$ and $Y = w\frac{d}{dt}$, we have the following:

$$\begin{aligned}
(\pounds_{\frac{d}{dt}} g_0)\left(v\frac{d}{dt}, w\frac{d}{dt}\right) &= \frac{d}{dt}\left(g_0\left(v\frac{d}{dt}, w\frac{d}{dt}\right)\right) - g_0\left(\left[\frac{d}{dt}, v\frac{d}{dt}\right], w\frac{d}{dt}\right) \\
&\quad - g_0\left(v\frac{d}{dt}, \left[\frac{d}{dt}, w\frac{d}{dt}\right]\right) \\
&= -v\frac{dw}{dt} - w\frac{dv}{dt} + w\frac{dv}{dt} - v\frac{dw}{dt} = 0; \\
(\pounds_{u\frac{d}{dt}} g_0)\left(v\frac{d}{dt}, w\frac{d}{dt}\right) &= u(\pounds_{\frac{d}{dt}} g_0)\left(v\frac{d}{dt}, w\frac{d}{dt}\right) + du\left(v\frac{d}{dt}\right)dt\left(w\frac{d}{dt}\right) \\
&\quad + du\left(w\frac{d}{dt}\right)dt\left(v\frac{d}{dt}\right) \\
&= -2vw\frac{du}{dt} = 2g_0\left(v\frac{d}{dt}, w\frac{d}{dt}\right)\frac{du}{dt}; \\
(\pounds_\xi \pounds_\xi g_0)(X, Y) &= \xi((\pounds_{u\frac{d}{dt}} g_0)(X, Y)) - (\pounds_{u\frac{d}{dt}} g_0)([\xi, X], Y) - (\pounds_{u\frac{d}{dt}} g_0)(X, [\xi, Y]) \\
&= -2u\left[\left(v\frac{dw}{dt} + w\frac{dv}{dt}\right)\frac{du}{dt} + vw\frac{d^2u}{dt^2}\right] \\
&\quad + 2uw\frac{du}{dt}\frac{dv}{dt} - 2vw\left(\frac{du}{dt}\right)^2 + 2uv\frac{du}{dt}\frac{dw}{dt} - 2vw\left(\frac{du}{dt}\right)^2 \\
&= -2vwu\frac{d^2u}{dt^2} - 4vw\left(\frac{du}{dt}\right)^2 \\
&= 2u\frac{d^2u}{dt^2} g_0\left(v\frac{d}{dt}, w\frac{d}{dt}\right) + 4\left(\frac{du}{dt}\right)^2 g_0\left(v\frac{d}{dt}, w\frac{d}{dt}\right). \quad \square
\end{aligned}$$

We can now conclude the following.

Proposition 5. *If $\xi = u\frac{d}{dt}$, with u being a smooth function on I, then, we have the following:*

(i) *ξ is a Killing vector field on (I, g_0) if and only if u is a constant;*

(ii) *ξ is a 2-Killing vector field on (I, g_0) if and only if*

$$u = \sqrt[3]{c_1 t + c_2}, \quad c_1, c_2 \in \mathbb{R}.$$

Proof. From Lemma 3, we deduce that ξ is a Killing vector field on (I, g_0) if and only if $\frac{du}{dt} = 0$, and ξ is a 2-Killing vector field on (I, g_0) if and only if $u\frac{d^2u}{dt^2} + 2\left(\frac{du}{dt}\right)^2 = 0$; hence, we have the conclusion. □

Now, we will provide the necessary and sufficient condition for a vector field on a doubly warped spacetime to be Killing or 2-Killing.

Theorem 6. *Let $\xi = u\frac{d}{dt} + \zeta$, with u being a smooth function on I and $\zeta \in \Gamma(TM)$. Then, we have the following:*

(i) ξ is a Killing vector field on the doubly warped spacetime ${}_{f_2}I \times_{f_1} M$ if and only if

$$\begin{cases} \frac{du}{dt} = -\frac{\zeta(f_2)}{f_2} \\ \pounds_\zeta g = -\frac{u}{f_1^2}\frac{df_1^2}{dt} g \end{cases};$$

(ii) ξ is a 2-Killing vector field on the doubly warped spacetime ${}_{f_2}I \times_{f_1} M$ if and only if

$$\begin{cases} 2\left[u\frac{d^2u}{dt^2} + 2\left(\frac{du}{dt}\right)^2\right] f_2^2 + 4\frac{du}{dt}\zeta(f_2^2) + \zeta(\zeta(f_2^2)) = 0 \\ f_1^2 \pounds_\zeta \pounds_\zeta g + 2u\frac{df_1^2}{dt}\pounds_\zeta g + u\left(u\frac{d^2f_1^2}{dt^2} + \frac{du}{dt}\frac{df_1^2}{dt}\right) g = 0 \end{cases}.$$

Proof. Let $X = X_1 + X_2, Y = Y_1 + Y_2 \in \Gamma(T({}_{f_2}I \times_{f_1} M))$. By replacing the expressions of $\pounds_\xi g_0$ and $\pounds_\xi \pounds_\xi g_0$ from Lemma 3 into Proposition 2, we consequently have

$$\begin{aligned} (\pounds_\xi \tilde{g})(X, Y) &= 2f_2^2\frac{du}{dt}g_0(X_1, Y_1) + \zeta(f_2^2)g_0(X_1, Y_1) \\ &\quad + f_1^2(\pounds_\zeta g)(X_2, Y_2) + u\frac{df_1^2}{dt}g(X_2, Y_2); \\ (\pounds_\xi \pounds_\xi \tilde{g})(X, Y) &= 2f_2^2\left[u\frac{d^2u}{dt^2} + 2\left(\frac{du}{dt}\right)^2\right]g_0(X_1, Y_1) + 4\zeta(f_2^2)\frac{du}{dt}g_0(X_1, Y_1) \\ &\quad + \zeta(\zeta(f_2^2))g_0(X_1, Y_1) + f_1^2(\pounds_\zeta \pounds_\zeta g)(X_2, Y_2) \\ &\quad + 2u\frac{df_1^2}{dt}(\pounds_\zeta g)(X_2, Y_2) + u\frac{d}{dt}\left(u\frac{df_1^2}{dt}\right)g(X_2, Y_2). \end{aligned}$$

If ξ is a Killing vector field, then

$$\begin{cases} 2\frac{du}{dt}f_2^2 + \zeta(f_2^2) = 0 \\ f_1^2\pounds_\zeta g + u\frac{df_1^2}{dt}g = 0 \end{cases},$$

which is equivalent to

$$\begin{cases} \frac{du}{dt} = -\frac{\zeta(f_2)}{f_2} \\ \pounds_\zeta g = -\frac{u}{f_1^2}\frac{df_1^2}{dt} g \end{cases}.$$

If ξ is a 2-Killing vector field, then

$$\begin{cases} 2\left[u\frac{d^2u}{dt^2} + 2\left(\frac{du}{dt}\right)^2\right] f_2^2 + 4\frac{du}{dt}\zeta(f_2^2) + \zeta(\zeta(f_2^2)) = 0 \\ f_1^2 \pounds_\zeta \pounds_\zeta g + 2u\frac{df_1^2}{dt}\pounds_\zeta g + u\frac{d}{dt}\left(u\frac{df_1^2}{dt}\right) g = 0 \end{cases},$$

and we obtain the conclusion. □

Example 2. *According to Theorem 6 (ii), the lift of any vector field $\zeta \in \Gamma(TM)$ to the doubly warped spacetime ${}_{f_2}I \times_{f_1} M$ is a 2-Killing vector field.*

In particular, we have the following results.

Corollary 5. *Let $\xi = u\frac{d}{dt}$, with $u \neq 0$ being a smooth function on I. Then, we have the following:*

(i) *the lift of ξ is a Killing vector field on the doubly warped spacetime ${}_{f_2}I \times_{f_1} M$ if and only if*

$$\begin{cases} \frac{du}{dt} = 0 \\ u\frac{df_1^2}{dt} = 0 \end{cases},$$

that is,

$$\begin{cases} u = c_1, \ \ c_1 \in \mathbb{R}^* \\ f_1 = c_2, \ \ c_2 \in \mathbb{R}_+^* \end{cases};$$

in this case, the manifold is a standard static spacetime;

(ii) *the lift of ξ is a 2-Killing vector field on the doubly warped spacetime ${}_{f_2}I \times_{f_1} M$ if and only if*

$$\begin{cases} u\frac{d^2u}{dt^2} + 2\left(\frac{du}{dt}\right)^2 = 0 \\ u\left(u\frac{d^2 f_1^2}{dt^2} + \frac{du}{dt}\frac{df_1^2}{dt}\right) = 0 \end{cases},$$

that is,

$$\begin{cases} u = \sqrt[3]{c_1 t + c_2}, \ \ c_1, c_2 \in \mathbb{R} \\ u\frac{df_1^2}{dt} = c_3, \ \ c_3 \in \mathbb{R} \end{cases};$$

in this case, $f_1^2(t) := c_4\sqrt[3]{(c_1 t + c_2)^2} + c_5$ for some $c_4, c_5 \in \mathbb{R}$.

Proof. It follows from Theorem 6. □

Example 3. *According to Corollary 5 (ii), the lift of the vector field $\xi = \sqrt[3]{c_1 t + c_2}\frac{d}{dt}$ on I, $c_1, c_2 \in \mathbb{R}^*$ to the doubly warped spacetime ${}_{f_2}I \times_{f_1} M$ is a 2-Killing vector field if and only if the warping function $f_1 : I \to (0, \infty)$ satisfies $f_1^2(t) := c_3\sqrt[3]{(c_1 t + c_2)^2} + c_4$ for some $c_3, c_4 \in \mathbb{R}$.*

Corollary 6. *Let $\xi = u\frac{d}{dt} + \zeta$, with u being a constant and $\zeta \in \Gamma(TM)$. Then, we have the following:*

(i) *ξ is a Killing vector field on the doubly warped spacetime ${}_{f_2}I \times_{f_1} M$ if and only if*

$$\begin{cases} \zeta(f_2) = 0 \\ \pounds_\zeta g = -\frac{u}{f_1^2}\frac{df_1^2}{dt} g \end{cases};$$

(ii) *ξ is a 2-Killing vector field on the doubly warped spacetime ${}_{f_2}I \times_{f_1} M$ if and only if*

$$\begin{cases} \zeta(\zeta(f_2^2)) = 0 \\ f_1^2 \pounds_\zeta \pounds_\zeta g + 2u\frac{df_1^2}{dt}\pounds_\zeta g + u^2\frac{d^2 f_1^2}{dt^2} g = 0 \end{cases}.$$

Proof. It follows from Theorem 6. □

Corollary 7. *Let $\xi = u\frac{d}{dt}$, with $u \neq 0$ being a constant. Then, we have the following:*

(i) *the lift of* ξ *is a Killing vector field on the doubly warped spacetime* ${}_{f_2}I\times_{f_1}M$ *if and only if*

$$f_1 = c, \;\; c \in \mathbb{R}_+^*,$$

that is, the manifold is a standard static spacetime;

(ii) *the lift of* ξ *is a 2-Killing vector field on the doubly warped spacetime* ${}_{f_2}I\times_{f_1}M$ *if and only if*

$$f_1(t) = \sqrt{c_1 t + c_2}, \;\; c_1, c_2 \in \mathbb{R} \;\; \textit{such that} \;\; c_1 t + c_2 > 0 \;\; \textit{for any} \;\; t \in I.$$

Proof. It follows from Corollary 5. □

Proposition 6. *Let* $\xi = u\frac{d}{dt} + \zeta$*, with* $u \neq 0$ *being a constant and* ζ *being a Killing vector field on* (M, g)*. Then, we have the following:*

(i) ξ *is a Killing vector field on the doubly warped spacetime* ${}_{f_2}I\times_{f_1}M$ *if and only if*

$$\begin{cases} \zeta(f_2) = 0 \\ f_1 = c, \;\; c \in \mathbb{R}_+^* \end{cases},$$

that is, f_2 *is constant on the integral curves of* ζ*, and the manifold is a standard static spacetime;*

(ii) ξ *is a 2-Killing vector field on the doubly warped spacetime* ${}_{f_2}I\times_{f_1}M$ *if and only if*

$$\begin{cases} \zeta(\zeta(f_2^2)) = 0 \\ f_1(t) = \sqrt{c_1 t + c_2}, \;\; c_1, c_2 \in \mathbb{R} \;\; \textit{such that} \;\; c_1 t + c_2 > 0 \;\; \textit{for any} \;\; t \in I \end{cases}.$$

Proof. It follows from Theorem 6. □

For the standard static spacetime, we deduce the following proposition.

Proposition 7. *If* $\xi = \zeta + u\frac{d}{dt}$*, with* $\zeta \in \Gamma(TM)$ *and* u *being a smooth function on* I*, then we have the following:*

(i) ξ *is a Killing vector field on the standard static spacetime* $M \times_f I$ *if and only if*

$$\begin{cases} \zeta(f) = -f\frac{du}{dt} \\ \pounds_\zeta g = 0 \end{cases};$$

(ii) ξ *is a 2-Killing vector field on the standard static spacetime* $M \times_f I$ *if and only if*

$$\begin{cases} 2\left[u\frac{d^2u}{dt^2} + 2\left(\frac{du}{dt}\right)^2\right]f^2 + 4\frac{du}{dt}\zeta(f^2) + \zeta(\zeta(f^2)) = 0 \\ \pounds_\zeta \pounds_\zeta g = 0 \end{cases}.$$

Moreover, if u *is a constant, then we have the following:*

(i') ξ *is a Killing vector field on the standard static spacetime* $M \times_f I$ *if and only if*

$$\begin{cases} \zeta(f) = 0 \\ \pounds_\zeta g = 0 \end{cases};$$

(ii') ξ *is a 2-Killing vector field on the standard static spacetime* $M \times_f I$ *if and only if*

$$\begin{cases} \zeta(\zeta(f^2)) = 0 \\ \pounds_\zeta \pounds_\zeta g = 0 \end{cases}.$$

Proof. It follows from Theorem 6. □

Remark 2. *Let $\xi = \zeta + u\frac{d}{dt}$, with $\zeta \in \Gamma(TM)$ and u being a smooth function on I. If ξ is a Killing vector field on $M \times_f I$, then ζ is a Killing vector field on (M, g), and if ξ is 2-Killing on $M \times_f I$, then ζ is also 2-Killing on (M, g).*

Corollary 8. *If $\xi = u\frac{d}{dt}$, with u being a smooth function on I, then we have the following:*

(i) the lift of ξ is a Killing vector field on the standard static spacetime $M \times_f I$ if and only if u is a constant;

(ii) the lift of ξ is a 2-Killing vector field on the standard static spacetime $M \times_f I$ if and only if

$$u = \sqrt[3]{c_1 t + c_2}, \ \ c_1, c_2 \in \mathbb{R}.$$

Moreover, if u is a constant, then the lift of ξ is a Killing, hence, a 2-Killing vector field on the standard static spacetime.

Proof. It follows from Proposition 7. □

For the generalized Robertson–Walker spacetime, we deduce the following proposition.

Proposition 8. *If $\xi = u\frac{d}{dt} + \zeta$, with u being a smooth function on I and $\zeta \in \Gamma(TM)$, then we have the following:*

(i) ξ is a Killing vector field on the generalized Robertson–Walker spacetime $I \times_f M$ if and only if

$$\begin{cases} u = c, \ \ c \in \mathbb{R} \\ \pounds_\zeta g = -u \cdot d(\ln f^2) g \end{cases};$$

(ii) ξ is a 2-Killing vector field on the generalized Robertson–Walker spacetime $I \times_f M$ if and only if

$$\begin{cases} u = \sqrt[3]{c_1 t + c_2}, \ \ c_1, c_2 \in \mathbb{R} \\ f^2 \pounds_\zeta \pounds_\zeta g + 2u\frac{df^2}{dt}\pounds_\zeta g + u\Big(u\frac{d^2 f^2}{dt^2} + \frac{du}{dt}\frac{df^2}{dt}\Big) g = 0 \end{cases}.$$

Proof. It follows from Theorem 6. □

Corollary 9. *If $\xi = u\frac{d}{dt}$, with $u \neq 0$ being a smooth function on I, then we have the following:*

(i) the lift of ξ is a Killing vector field on the generalized Robertson–Walker spacetime $I \times_f M$ if and only if

$$\begin{cases} u = c_1, \ \ c_1 \in \mathbb{R}^* \\ f = c_2, \ \ c_2 \in \mathbb{R}^*_+ \end{cases},$$

that is, the manifold is a direct product;

(ii) the lift of ξ is a 2-Killing vector field on the generalized Robertson–Walker spacetime $I \times_f M$ if and only if

$$\begin{cases} u = \sqrt[3]{c_1 t + c_2}, \ \ c_1, c_2 \in \mathbb{R} \\ u\frac{df^2}{dt} = c_3, \ \ c_3 \in \mathbb{R}. \end{cases}$$

Moreover, if u is a constant, then the lift of ξ is a Killing or a 2-Killing vector field on the generalized Robertson–Walker spacetime $I \times_f M$ if and only if

$$f = c, \ \ c \in \mathbb{R}^*_+,$$

that is, the manifold is a direct product.

Proof. It follows from Proposition 8. □

Example 4. *We consider the generalized Robertson–Walker spacetime $I \times_f M$, with $f : I \to \mathbb{R}_+^*$ and $f(t) := \sqrt[3]{t}$. Then, according to Corollary 9 (ii), the lift of the vector field $\xi = c\sqrt[3]{t}\frac{d}{dt}$ on I, where $c \in \mathbb{R}^*$, to $I \times_f M$ is a 2-Killing vector field.*

5. Conclusions

Distinguished vector fields on a Riemannian manifold dictate some of its geometrical properties. In the present paper, we have continued the studies concerning the Killing and 2-Killing vector fields on warped product and doubly warped product manifolds [8,9], thereby bringing to light new properties of them. More precisely, we have provided a condition for a more general vector field than a 2-Killing vector field on a compact manifold to be parallel, we have established some consequences on the factor manifolds when the doubly warped product is endowed with a 2-Killing vector field, we have provided the necessary and sufficient conditions for a doubly warped product to reduce to a direct product when there exists a 2-Killing vector field on the factor manifolds, and we have shown that, under a certain assumption, a factor manifold is a Ricci or a hyperbolic Ricci soliton with a potential vector field that is the component of a Killing or of a 2-Killing vector field. Regarding physical applications, we considered the spacetime case, and, in particular, the standard static spacetime and the generalized Robertson–Walker spacetime. This study has been continued by the present authors for the multiply warped product and multiply twisted product spacetimes.

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