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The Inverse Weber Problem on the Plane and the Sphere

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Abstract: Weber's inverse problem in the plane is to modify the positive weights associated with n fixed points in the plane at minimum cost, ensuring that a given point a priori becomes the Euclidean weighted geometric median. In this paper, we investigate Weber's inverse problem in the plane and generalize it to the surface of the sphere. Our study uses a subspace orthogonal to a subspace generated by two vectors X and Y associated with the given points and weights. The main achievement of our work lies in determining a vector perpendicular to the vectors X and Y, in $\mathbb{R}^n$; which is used to determinate a solution of Weber's inverse problem. In addition, lower bounds are obtained for the minimum of the Weber function, and an upper bound for the difference of the minimal of Weber's direct and inverse problems. Examples of application at the plane and unit sphere are given.

Keywords: Weber's problem; Weber's inverse problem; location of services; orthogonal space

MSC: 46N10; 47N10; 15A29

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1. Introduction

Inverse problems appear in different areas of science; that is why in recent decades the study of such problems has gained interest. Inverse problems are usually ill-posed problems, as some of the conditions given by Hadamard [1] for a well-posed problem are not fulfilled.

In optimization, Burton and Toint [2], motivated by practical situations, studied the inverse shortest paths problem in a graph, formulating an algorithm based on the method of Goldfarb–Idnani. In location theory, Berman et al. [3] studied the inverse 1-median problem over a network. In 1999, Cai et al. [4] studied the inverse center location problem. Zhan, Liu, and Ma [5] studied the inverse center location problem on a tree with equal weights associated with the vertices.

In 2004, Burkard, Pleschiutschnig, and Zhang [6] studied different inverse median problems. In 2010, Burkard, Galavii, and Gassner [7] studied the inverse Fermat–Weber problem in the plane, presenting a purely combinatorial $O(nlogn)$ algorithm for this problem.

In this paper, we investigate Weber's inverse problem in the plane, and generalize it to the surface of the sphere, which is used to model the planet Earth. The sphere, as a regular surface, has its own metric, characterized by the intrinsic distance or geodesic distance.

Our study differs from the approach of Burkard et al. [7], in that our analysis is given in the orthogonal subspace $\sigma^{\perp}$ to the vector subspace σ generated by the vectors X and Y obtained from the given points and weights. It is assumed that the point given a priori is different from the given n points, and belongs to the interior of the convex capsule of those points.

Since the vector product of two vectors in $\mathbb{R}^n$ is not defined for $n > 3$, the achievement of our research is to obtain a vector perpendicular to the vectors X and Y in $\mathbb{R}^n$, which is used to determine a solution to Weber's inverse problem.

In addition, lower bounds are obtained for the minimum of the Weber function, and an upper bound for the difference of the minima of the Weber functions of the direct and inverse problems.

This paper is organized as follows: in Section 2 results are given on the direct and inverse Weber problems in the plane (Burkard et al. [7]). In Section 3 results are given on the Weber problem on the sphere (Drezner and Wesolowsky [8], Mangalika [9]). In Section 4 the theory is developed, and in Section 5 the inverse Weber problem is generalized to the sphere. Numerical examples in localization theory in the plane and on the sphere are given in Section 6.

2. Weber Problems: Direct and Inverse in the Plane

Given n points $p_i = (x_i, y_i) \in \mathbb{R}^2$, together with non-negative weights $w_i \in \mathbb{R}, i = 1, \ldots, n$; the Weber problem consists in finding a point $p_0 = (x_0, y_0) \in \mathbb{R}^2$ that minimizes the Weber function

$$F(p) = \sum_{i=1}^{n} w_i d(p, p_i), \tag{1}$$

where $d(p, p_i)$ is the Euclidean distance from point p to point p_i. The point $p_0 \in \mathbb{R}^2$ that minimizes (1) is called weighted geometric median (Fletcher et al. [10]).

The origin of this problem is attributed to the mathematician Pierre de Fermat, and several solutions were proposed. Evangelista Torricelli approached the problem with three points, and later Simpson [11] presented additional solutions to this problem, initially known as Fermat's problem.

Weiszfeld [12] proposed an iterative method that allows obtaining a point that minimizes the Weber function. For this purpose, he constructs a sequence given by

$$p_{k+1} = \frac{\displaystyle\sum_{i=1}^{n} \frac{w_i}{d(p_k, p_i)} p_i}{\displaystyle\sum_{i=1}^{n} \frac{w_i}{d(p_k, p_i)}}. \tag{2}$$

Definition 1 (Burkard [7])**.** *If $p_0 \neq p_i$, for all $i = 1, \ldots, n$, the resultant force $R(p_0)$ at p_0 is given by*

$$R(p_0) = \sum_{i=1}^{n} \frac{w_i}{d(p_0, p_i)} (p_i - p_0). \tag{3}$$

If $p_0 = p_j$ for some $j = 1, 2, \ldots, n$, we have

$$R(p_0) = \max\{\|R_j\| - w_j, 0\} \frac{R_j}{\|R_j\|},$$

where

$$R_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{w_i}{d(p_i, p_j)} (p_i - p_j).$$

Thus, for $p_0 = p_j$, we have $R(p_j) = 0$, if $w_j \geq \|R_j\|$.

Theorem 1. *The point p_0 is a solution of the Fermat–Weber problem if and only if $R(p_0) = 0$.*

For a proof, see, e.g., Kuhn [13].

Theorem 2 (Burkard [7])**.** *If the point p_0 is an optimal solution of the Fermat–Weber problem, then p_0 lies in the convex hull of the points $p_i, i = 1, \ldots, n$.*

Now, consider $n+1$ points $p_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 0, 1, \ldots, n$, with weights $w_i > 0, \forall i = 1, \ldots, n$.

The Inverse Weber problem in the plane consists in modifying the weights w_i with a minimum cost, in such a way that the point p_0 given a priori is the weighted geometric median, with respect to the new positive weights $w_i^*, i = 1, \ldots, n$.

To guarantee a finite solution, positive bounds $\underline{w}_i$ and $\overline{w}_i$ were considered such that

$$\underline{w}_i \le w_i \le \overline{w}_i, \forall i = 1, \ldots, n.$$

Thus, the inverse Weber problem in the plane can be posed as a linear programming problem, with $2n$ bounded variables and 2 equality constraints, Plastria [14], cited by Burkard et al. [7].

$$\begin{aligned} \text{Minimize } & \sum_{i=1}^{n} c_i(r_i + t_i) \\ \text{Subject to } & \\ & \sum_{i=1}^{n} (w_i + (r_i - t_i))\overline{x}_i = 0 \\ & \sum_{i=1}^{n} (w_i + (r_i - t_i))\overline{y}_i = 0 \\ & r_i \le \overline{w}_i - w_i\,, \forall i = 1, \ldots, n \\ & t_i \le w_i - \underline{w}_i\,, \forall i = 1, \ldots, n \\ & r_i, t_i \ge 0, \forall i = 1, \ldots, n, \end{aligned} \tag{4}$$

where $\sum_{i=1}^{n} c_i(r_i + t_i)$ is the total cost, r_i and t_i denote the amount at which w_i increases and decreases, respectively. This problem can be solved in linear time due to Megiddo [15].

Burkard et al. [6] made a study on the inverse median problem, and in 2010, in their paper on the Inverse Fermat–Weber problem on the plane, they presented an algorithm that runs in $O(nlogn)$ time.

Theorem 3 (Burkard [7])**.** *If p_0 lies in the interior of the convex hull of the points $p_i, i = 1, \ldots, n$, and the given bounds of the weights allow a feasible solution, then there always exists an optimal solution of the inverse Fermat–Weber problem, where at most two modified weights lie strictly between their lower and their upper bound.*

3. Weber's Problem in the Unit Sphere

In this section we give some results on the Weber problem on the unit Sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 1\}.$$

Given n different points $p_i \in S^2, i = 1, \ldots, n$, and weights $w_i > 0, \forall i = 1, \ldots, n$, the classical Weber problem consists of finding a point $p_0 \in S^2$ that minimizes the Weber function

$$F(p) = \sum_{i=1}^{n} w_i d_{S^2}(p, p_i), \tag{5}$$

where $d_{S^2}(p, p_i)$ is the intrinsic or geodesic distance from point p to point p_i.

Analogous to the case in the plane, the point p_0 is called weighted geometric median. This problem was initially studied by Drezner and Wesolowsky [16], and by Kats and Cooper [17]. On the sphere, the Weber function is nonconvex, which complicates the problem. Drezner [18] continued the study of this problem, and in 1983 Drezner and Wesolowsky [8] studied the minimax and maximin problem on the sphere. Hansen et al. [19] proposed an algorithm to approximate the solution to the Weber problem on the sphere, taking as a reference the study of the minisum and minimax problems [20].

Since the sphere is a regular surface, we can parameterize it using spherical coordinates given by the application $X : \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle \times \langle 0, 2\pi \rangle \to S^2$, defined by

$$X(\phi, \theta) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\theta).$$

Thus, the points are given by $p_i = X(\phi_i, \theta_i), i = 1, \ldots, n$.

In spherical coordinates, Weber's function (5) is given by

$$F(\phi,\theta) = \sum_{i=1}^{n} w_i d_{S^2}(X(\phi,\theta), X(\phi_i,\theta_i)). \tag{6}$$

The intrinsic distance or geodesic distance between points p and p_i is given by the length of the shortest arc of the maximum circle passing through these point

Theorem 4. *Given two points $X_1 = X(\phi_1,\theta_1)$, $X_2 = X(\phi_2,\theta_2) \in S^2$, the shortest arc length $\alpha = \alpha(X_1, X_2)$ verifies*

$$\sin\left(\frac{\alpha}{2}\right) = \frac{1}{2}\sqrt{2 - 2\sin\phi_1 \sin\phi_2 \cos(\theta_1 - \theta_2)}. \tag{7}$$

Proof. Using spherical coordinates, we have

$$\begin{aligned} X_1 = X(\phi_1,\theta_1) &= (\cos\phi_1\cos\theta_1, \cos\phi_1\sin\theta_1, \sin\phi_1), \\ X_2 = X(\phi_2,\theta_2) &= (\cos\phi_2\cos\theta_2, \cos\phi_2\sin\theta_2, \sin\phi_2). \end{aligned}$$

From Figure 1, we have

$$\|\overline{MX_2}\| = \frac{\|X_1 - X_2\|}{2},$$

then

$$\sin\left(\frac{\alpha}{2}\right) = \frac{\|X_1 - X_2\|}{2}. \tag{8}$$

Also

$$\|X_1 - X_2\|^2 = \|X(\phi_1,\theta_1) - X(\phi_2,\theta_2)\|^2 = 2 - 2\sin\phi_1\sin\phi_2\cos(\theta_1 - \theta_2).$$

Therefore
$\sin\left(\frac{\alpha}{2}\right) = \frac{1}{2}\sqrt{2 - 2\sin\phi_1 \sin\phi_2 \cos(\theta_1 - \theta_2)}$. □

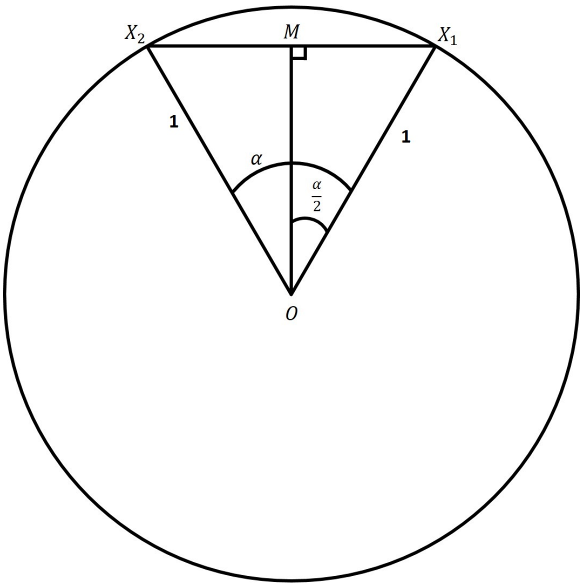

Figure 1. Great Circle on unit sphere.

Theorem 5 (Drezner and Wesolowsky [16], Mangalika [9])**.** *Let $X \in S^2$. The spherical circle $D(X, \frac{\pi}{2})$ is a convex set. The function $f : D(X, \frac{\pi}{2}) \to \mathbb{R}$ defined by $f(Y) = d_{S^2}(X,Y)$, $\forall Y \in D(X, \frac{\pi}{2})$ is a convex function.*

Theorem 6 (Drezner, Z. and Wesolowsky, G.O [16], Mangalika, D. [9])**.** *Let $X_i \in D(X_0, \frac{\pi}{4}) \subset S^2$, $i = 1, \ldots, n$. Then the Weber function $f : D(X_0, \frac{\pi}{4}) \to \mathbb{R}$ is a convex function and is minimized at a single point of $D(X_0, \frac{\pi}{4})$.*

4. Theory Development

In this section we construct a new method to find a solution to Weber's Inverse problem in the plane.

Consider $n+1$ points $p_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 0, 1, \ldots, n$, with weights $w_i > 0$, $\forall i = 1, \ldots, n$, and $p_0 \neq p_i$.

By (3) and Theorem 1, $p_0 = (x_0, y_0)$ will be the weighted geometric median, if and only if

$$R(p_0) = \sum_{i=1}^{n} \frac{w_i}{d(p_0, p_i)}(p_i - p_0) = 0.$$

Then

$$\sum_{i=1}^{n} \frac{w_i}{d_i}(x_i - x_0) = 0, \tag{9}$$

$$\sum_{i=1}^{n} \frac{w_i}{d_i}(y_i - y_0) = 0, \tag{10}$$

where $d_i = d(p_0, p_i)$, which is the Euclidean distance from point p_0 to point p_i.

By (9) and (10) we have

$$\langle (w_1, w_2, \ldots, w_n), \left(\frac{x_1 - x_0}{d_1}, \frac{x_2 - x_0}{d_2}, \ldots, \frac{x_n - x_0}{d_n} \right) \rangle = 0.$$

$$\langle (w_1, w_2, \ldots, w_n), \left(\frac{y_1 - y_0}{d_1}, \frac{y_2 - y_0}{d_2}, \ldots, \frac{y_n - y_0}{d_n} \right) \rangle = 0.$$

Let

$$w = (w_1, w_2, \ldots, w_n) \tag{11}$$

$$X = \left(\frac{x_1 - x_0}{d_1}, \frac{x_2 - x_0}{d_2}, \ldots, \frac{x_n - x_0}{d_n} \right), \tag{12}$$

$$Y = \left(\frac{y_1 - y_0}{d_1}, \frac{y_2 - y_0}{d_2}, \ldots, \frac{y_n - y_0}{d_n} \right), \tag{13}$$

it is verified:

$$\langle w, X \rangle = 0, \text{and } \langle w, Y \rangle = 0; \quad \text{that is } w \perp X \text{ and } w \perp Y. \tag{14}$$

Let $\sigma = \mathcal{L}(\{X, Y\})$ be the vector subspace generated by the vectors X and Y, and $\sigma^{\perp}$ the subspace orthogonal to σ. Then, from (14) it follows that the solutions to Weber's Inverse problem lie in the orthogonal subspace $\sigma^{\perp}$.

Let $N = (N_1, \ldots, N_n) \in \sigma^{\perp}$, $N \neq 0$. We consider the vector

$$w^* = (w_1^*, \ldots, w_n^*) = proj_N w. \tag{15}$$

Then

$$w^* = \frac{\langle w, N \rangle}{\|N\|^2} N.$$

That is

$$w_i^* = \frac{\langle w, N \rangle}{\|N\|^2} N_i, \quad \forall i = 1, \ldots, n. \tag{16}$$

Theorem 7. *Let $N = (N_1, \ldots, N_n) \in \sigma^{\perp}$, $N \neq 0$. If $N_i > 0$, $\forall i = 1, \ldots, n$, then Weber's Inverse problem has a solution given by*

$$w_i^* = \frac{\langle w, N \rangle}{\|N\|^2} N_i > 0, \quad \forall i = 1, \ldots, n. \tag{17}$$

Moreover, the angle $\theta = \measuredangle(w, N)$, verifies $\theta \in (0, \frac{\pi}{2})$.

Proof. Since $w_i > 0$ and $N_i > 0$, $\forall i = 1, \ldots, n$, it follows that $\langle w, N\rangle > 0$. Then

$$w_i^* = \frac{\langle w, N\rangle}{\|N\|^2} N_i > 0, \quad \forall i = 1, \ldots, n.$$

Moreover $\cos(\theta) = \frac{\langle w,N\rangle}{\|w\|\|N\|}$, then $\theta \in (0, \frac{\pi}{2})$. □

Theorem 8. *Let $N = (N_1, \ldots, N_n) \in \sigma^{\perp}$, $N \neq 0$. If $N_i < 0$, $\forall i = 1, \ldots, n$, then Weber's Inverse problem has a solution given by:*

$$w_i^* = \frac{\langle w, N\rangle}{\|N\|^2} N_i > 0, \quad \forall i = 1, \ldots, n. \tag{18}$$

Moreover, the angle $\theta = \measuredangle(w, N)$, verifies $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$.

Proof. Since $w_i > 0$ and $N_i < 0$, $\forall i = 1, \ldots, n$, it follows that $\langle w, N\rangle < 0$. Then

$$w_i^* = \frac{\langle w, N\rangle}{\|N\|^2} N_i > 0, \quad \forall i = 1, \ldots, n.$$

Moreover $\cos(\theta) = \frac{\langle w,N\rangle}{\|w\|\|N\|}$, then $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$. □

The following theorem shows that if there exist at least two components of N with different sign, the Weber inverse problem has no positive real solutions.

Theorem 9. *Suppose that $N_i > 0$, except some $i_0 \in \{1, \ldots, n\}$. Then:*

$$w_{i_0}^* < 0 \quad \text{and} \quad w_i^* > 0, \text{ for } i \neq i_0. \tag{19}$$
$$w_{i_0}^* > 0 \quad \text{and} \quad w_i^* < 0, \text{ for } i \neq i_0. \tag{20}$$

Proof. Without loss of generality we can assume that $i_0 = 1$, and

$$N_1 < 0 \text{ and } N_i > 0, \ \forall i = 2, \ldots, n.$$

There are two cases:
If $\sum_{i=1}^{n} w_i N_i > 0$, then $\langle w, N\rangle > 0$. Therefore

$$w_1^* = \frac{\langle w, N\rangle}{\|N\|^2} N_1 < 0, \quad \text{and } w_i^* = \frac{\langle w, N\rangle}{\|N\|^2} N_i > 0, \quad \forall i = 2, \ldots, n,$$

which proves (19).
If $\sum_{i=1}^{n} w_i N_i < 0$, then $\langle w, N\rangle < 0$. Therefore

$$w_1^* = \frac{\langle w, N\rangle}{\|N\|^2} N_1 > 0, \quad \text{and } w_i^* = \frac{\langle w, N\rangle}{\|N\|^2} N_i < 0, \quad \forall i = 2, \ldots, n,$$

which proves (20). □

Next, using orthogonality properties, we obtain a vector orthogonal to two linearly independent vectors in $\mathbb{R}^n$.

Let $X, Y \in \mathbb{R}^n$ be linearly independent, $\sigma = \mathcal{L}(\{X, Y\})$ be the vector subspace generated by X and Y, $\sigma^{\perp}$ the orthogonal complement to σ, $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, $w \notin \sigma$. Then

$$w = w^T + w^N, \tag{21}$$

where

- w^T is the orthogonal projection of w onto σ,

- w^N is the orthogonal projection of w onto $\sigma^\perp$.
  Since $\{X, Y\}$ is a basis of σ, then

$$w^T = aX + bY, \quad a, b \in \mathbb{R}, \tag{22}$$

and since $w^N \in \sigma^\perp$, then

$$\begin{aligned} \langle w^N, X \rangle &= 0, & (23) \\ \langle w^N, Y \rangle &= 0, & (24) \end{aligned}$$

Theorem 10. *Let $X, Y \in \mathbb{R}^n$ be linearly independent, $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, $w^T = aX + bY$. Then:*

$$\begin{aligned} a &= \langle w, \frac{X{<}Y,Y{>} - Y{<}X,Y{>}}{\Delta}, \rangle, & (25) \\ b &= \langle w, \frac{Y{<}X,X{>} - X{<}X,Y{>}}{\Delta}, \rangle, & (26) \end{aligned}$$

where

$$\Delta = \|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2. \tag{27}$$

Proof. From (23) we have $\langle w^N, X \rangle = 0$. Further from (21) we have $w^N = w - w^T$, then

$$w^N = w - aX - bY.$$

Therefore, we have

$$0 = \langle w^N, X \rangle = \langle w - aX - bY, X \rangle = \langle w, X \rangle - a\langle X, X \rangle - b\langle X, Y \rangle.$$

Then

$$a\langle X, X \rangle + b\langle X, Y \rangle = \langle w, X \rangle \tag{28}$$

Similarly, by (24) we obtain

$$a\langle X, Y \rangle + b\langle Y, Y \rangle = \langle w, Y \rangle \tag{29}$$

By (28) and (29) we obtain the matrix system

$$\begin{pmatrix} \langle X, X \rangle & \langle X, Y \rangle \\ \langle X, Y \rangle & \langle Y, Y \rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \langle w, X \rangle \\ \langle w, Y \rangle \end{pmatrix}. \tag{30}$$

The determinant of the system (30) is

$$\Delta = \|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2 \neq 0.$$

Using Cramer's rule we obtain

$$\begin{aligned} a &= \langle w, \frac{X{<}Y,Y{>} - Y{<}X,Y{>}}{\triangle} \rangle, \\ b &= \langle w, \frac{Y{<}X,X{>} - X{<}X,Y{>}}{\triangle} \rangle. \end{aligned}$$

which proves the theorem. □

Therefore, a normal vector to X and Y is given by:

$$w^N = w - aX - bY.$$

Now, let $p_i = (x_i, y_i) \in \mathbb{R}^2$, $\forall i = 1, \ldots, n$ be different non-collinear points, $w_i > 0$, $\forall i = 1, \ldots, n$ the weights, and $p_0^* = (x_0^*, y_0^*)$ an interior point of the convex capsule of the $p_i, i = 1, \ldots, n$.

By (12) and (13) we have the vectors

$$X = \left(\frac{x_1 - x_0^*}{d_1}, \ldots, \frac{x_n - x_0^*}{d_n}\right) = (X_1, \ldots, X_n),$$
$$Y = \left(\frac{y_1 - y_0^*}{d_1}, \ldots, \frac{y_n - y_0^*}{d_n}\right) = (Y_1, \ldots, Y_n),$$

where

$$d_i = d_i(p_0^*, p_i), \; \forall i = 1, \ldots, n.$$

Theorem 11. *Let $w^N = (w_1^N, \ldots, w_n^N) \in \sigma^\perp$, we have*

$$w_i^N > 0, \; \forall i = 1, \ldots, n, \quad \textit{if and only if} \;\; w_i > \langle w, R_i \rangle, \; \forall i = 1, \ldots, n. \tag{31}$$

where

$$R_i = \frac{[X<Y,Y> - Y<X,Y>]X_i + [Y<X,X> - X<X,Y>]Y_i}{\triangle}, \; \forall i = 1, \ldots, n. \tag{32}$$

Proof. If $w_i^N > 0, \quad \forall i = 1, \ldots, n$, then $w_i^N = w_i - aX_i - bY_i > 0, \; \forall i = 1, \ldots, n$

$$\begin{aligned} w_i &> aX_i + bY_i \\ &= \langle w, \frac{X<Y,Y> - Y<X,Y>}{\triangle} \rangle X_i + \langle w, \frac{Y<X,X> - X<X,Y>}{\triangle} \rangle Y_i, \\ &= \langle w, \frac{[X<Y,Y> - Y<X,Y>]X_i}{\triangle} \rangle + \langle w, \frac{[Y<X,X> - X<X,Y>]Y_i}{\triangle} \rangle, \\ &= \langle w, \frac{[X<Y,Y> - Y<X,Y>]X_i}{\triangle} + \frac{[Y<X,X> - X<X,Y>]Y_i}{\triangle} \rangle. \end{aligned}$$

Therefore

$$w_i > \langle w, R_i \rangle, \forall i = 1, \ldots, n,$$

where

$$R_i = \frac{[X<Y,Y> - Y<X,Y>]X_i + [Y<X,X> - X<X,Y>]Y_i}{\triangle}, \; \forall \, i = 1, \ldots, n.$$

Reciprocally, if $w_i > \langle w, R_i \rangle, \; \forall i = 1, \ldots, n$, then

$$\begin{aligned} <w, R_i> &= \langle w, \frac{[X<Y,Y> - Y<X,Y>]X_i}{\triangle} + \frac{[Y<X,X> - X<X,Y>]Y_i}{\triangle} \rangle > \\ &= \langle w, \frac{[X<Y,Y> - Y<X,Y>]X_i}{\triangle} \rangle + \langle w, \frac{[Y<X,X> - X<X,Y>]Y_i}{\triangle} \rangle. \end{aligned}$$

Then

$$w_i > aX_i + bY_i, \quad \forall i = 1, \ldots, n.$$

Therefore, we have
$w_i^N = w_i - aX_i - bY_i > 0, \quad \forall i = 1, \ldots, n.$ □

Theorem 12. *Let $w_i > \langle w, R_i \rangle, \; \forall i = 1, \ldots, n$, p_0 the weighted geometric median of the Weber function, then:*

$$F(p) \;\; > \;\; \langle w, \sum_{i=1}^{n} R_i d_i(p, p_i) \rangle, \quad \forall p \in \mathbb{R}^2. \tag{33}$$

$$F(p_0) \;\; > \;\; \langle w, \sum_{i=1}^{n} R_i d_i(p_0, p_i) \rangle, \tag{34}$$

where R_i is given in (32).

Proof. As $w_i > \langle w, R_i\rangle$,$\forall i = 1, \dots, n$, then $w_i d_i(p, p_i) > \langle w, R_i\rangle d_i(p, p_i), \forall i = 1, \dots, n$, following $\sum_{i=1}^{n} w_i d_i(p, p_i) > \sum_{i=1}^{n}\langle w, R_i\rangle d_i(p, p_i)$.

Therefore

$F(p) > \langle w, \sum_{i=1}^{n} R_i d_i(p, p_i)\rangle, \forall p \in \mathbb{R}^2$, which proves (33).

For (34), we see that if p_0 is the weighted geometric median, then $F(p_0) = \min_{p\in\mathbb{R}^2} F(p) > \langle w, \sum_{i=1}^{n} R_i d_i(p_0, p_i)\rangle$. □

Note that the above theorem allows us to obtain lower bounds for the minimum of the Weber function.

Theorem 13. *Let $p_i = (x_i, y_i) \in \mathbb{R}^2, \forall i = 1, \dots, n$ be different non-collinear points, $w_i > 0$, $\forall i = 1, \dots, n$ the weights, and $p_0^* = (x_0^*, y_0^*)$ an interior point of the convex capsule of the p_i.*

If $w_i > \langle w, R_i\rangle$, $\forall i = 1, \dots, n$, then the Inverse Weber problem in the plane has a solution given by

$$w_i^* = w_i - aX_i - bY_i, \quad \forall i = 1, \dots, n. \tag{35}$$

where

$$X_i = \frac{x_i - x_0^*}{d_i(p_0^*, p_i)} \quad \textit{and} \quad Y_i = \frac{y_i - y_0^*}{d_i(p_0^*, p_i)}.$$

Proof. By using the points $p_i = (x_i, y_i)$ and $p_0^* = (x_0^*, y_0^*)$ we obtain the linearly independent vectors

$$X = \left(\frac{x_1 - x_0^*}{d_1}, \dots, \frac{x_n - x_0^*}{d_n}\right) \quad \text{and} \quad Y = \left(\frac{y_1 - y_0^*}{d_1}, \dots, \frac{y_n - y_0^*}{d_n}\right).$$

Moreover, by construction, the vector $w^N = w - aX - bY$ is orthogonal to the vectors X and Y; and since $w_i > \langle w, R_i\rangle$,$\forall i = 1, \dots, n$, we have that

$$w_i^N > 0, \quad i = 1, \dots, n.$$

Therefore, a solution to Weber's inverse problem is given by $w_i^* = w_i^N, \quad \forall i = 1, \dots, n.$ □

Theorem 14. *With the assumptions of the previous theorem we have*

$$R(p_0^*) = 0 \quad \textit{if and only if} \quad \langle w^*, X\rangle = 0, \quad \langle w^*, Y\rangle = 0, \tag{36}$$

Proof.

$$\begin{aligned} 0 = R(p_0^*) &= \sum_{i=1}^{n} \frac{w_i^*(p_i - p_0^*)}{d(p_0^*, p_i)} = \left(\sum_{i=1}^{n} \frac{w_i^*(x_i - x_0^*)}{d(p_0^*, p_i)}, \sum_{i=1}^{n} \frac{w_i^*(y_i - y_0^*)}{d(p_0^*, p_i)}\right) \\ &= (\sum_{i=1}^{n} w_i^* X_i, \sum_{i=1}^{n} w_i^* Y_i) = (0, 0) \end{aligned}$$

Therefore, we have

$$\langle w^*, X\rangle = 0, \quad \langle w^*, Y\rangle = 0.$$

Reciprocally, if $0 = \langle w^*, X\rangle = \sum_{i=1}^{n} w_i^* X_i = \sum_{i=1}^{n} \frac{w_i^*(x_i - x_0^*)}{d(p_0^*, p_i)} = R_x(p_0^*)$, and $0 = \langle w^*, Y\rangle = \sum_{i=1}^{n} w_i^* Y_i = \sum_{i=1}^{n} \frac{w_i^*(y_i - y_0^*)}{d(p_0^*, p_i)} = R_y(p_0^*)$.

Therefore $R(p_0^*) = (R_x(p_0^*), R_y(p_0^*)) = (0, 0)$. □

Theorem 15. *Let* $F(p) = \sum_{i=1}^{n} w_i d(p, p_i)$ *and* $F^*(p) = \sum_{i=1}^{n} w_i^* d(p, p_i)$ *the Weber functions, and* $w_i^N > 0$, $\forall i = 1, \ldots, n$, *then*

$$F^*(p) = F(p) - a\sum_{i=1}^{n} X_i d(p, p_i) - b\sum_{i=1}^{n} Y_i d(p, p_i), \tag{37}$$

$$F(p_0) = \min_{p\in\mathbb{R}^2} F(p) > \sum_{i=1}^{n}(aX_i + bY_i)d(p_0, p_i). \tag{38}$$

Proof. We have that $w_i^N > 0$, $\forall i = 1, \ldots, n$, then

$$\begin{aligned} w_i^* &= w_i - aX_i - bY_i, \\ w_i^* d(p, p_i) &= w_i d(p, p_i) - aX_i d(p, p_i) - bY_i d(p, p_i). \\ \sum_{i=1}^{n} w_i^* d(p, p_i) &= \sum_{i=1}^{n} w_i d(p, p_i) - a\sum_{i=1}^{n} X_i d(p, p_i) - b\sum_{i=1}^{n} Y_i d(p, p_i). \end{aligned}$$

Therefore we have the proof of (37)

$$F^*(p) = F(p) - a\sum_{i=1}^{n} X_i d(p, p_i) - b\sum_{i=1}^{n} Y_i d(p, p_i).$$

In order to prove (38), if p_0 is the weighted geometric median of F, by (37) we have

$$F^*(p_0) = F(p_0) - a\sum_{i=1}^{n} X_i d(p_0, p_i) - b\sum_{i=1}^{n} Y_i d(p_0, p_i) > 0.$$

Then $F(p_0) > \sum_{i=1}^{n}(aX_i + bY_i)d(p_0, p_i)$. □

Theorem 16. *Let* $p_i = (x_i, y_i) \in \mathbb{R}^2$, $\forall i = 1, \ldots, n$ *be different non-collinear points,* p_0 *and* p_0^* *the weighted geometric median of the Weber functions* F *and* F^* *respectively, and* D *the convex capsule of the points* p_i.

If $w_i^N > 0$, $\forall i = 1, \ldots, n$, *then*

$$|F^*(p) - F(p)| \le \big(|a|S_x + |b|S_y\big)diam(D), \ \forall p \in D, \tag{39}$$

where

$S_x = \sum_{i=1}^{n} |X_i|$, $S_y = \sum_{i=1}^{n} |Y_i|$, *and* $diam(D)$ *is the diameter of the set* D.

Proof. Since $w_i^N > 0$, $\forall i = 1, \ldots, n$, then by (37) we have

$$\begin{aligned} |F^*(p) - F(p)| &= \Big| - a\sum_{i=1}^{n} X_i d(p, p_i) - b\sum_{i=1}^{n} Y_i d(p, p_i)\Big| \\ &= \Big| - \sum_{i=1}^{n}(aX_i + bY_i)d(p, p_i)\Big| \\ &\le \sum_{i=1}^{n}(|a||X_i| + |b||Y_i|)d(p, p_i) \\ &= |a|\sum_{i=1}^{n} |X_i| d(p, p_i) + |b|\sum_{i=1}^{n} |Y_i| d(p, p_i). \end{aligned}$$

Now, if $p \in D$, then $d(p, p_i) \le diam(D)$. Therefore

$$|F^*(p) - F(p)| \le (|a|S_x + |b|S_y)diam(D), \quad \forall p \in D,$$

where $S_x = \sum_{i=1}^{n} |X_i|$ and $S_y = \sum_{i=1}^{n} |Y_i|$. $\square$

Theorem 17. *With the assumptions of the theorem (16) we have*

$$|F^*(p_0^*) - F(p_0)| \leq |F(p_0^*) - F^*(p_0)| + 2diam(D)(|a|S_x + |b|S_y), \quad \forall p \in D. \tag{40}$$

Proof. By (37):

$$F^*(p) = F(p) - a\sum_{i=1}^{n} X_i d(p, p_i) - b\sum_{i=1}^{n} Y_i d(p, p_i),$$

then

$$F^*(p_0^*) = F(p_0^*) - a\sum_{i=1}^{n} X_i d(p_0^*, p_i) - b\sum_{i=1}^{n} Y_i d(p_0^*, p_i), \tag{41}$$

$$F^*(p_0) = F(p_0) - a\sum_{i=1}^{n} X_i d(p_0, p_i) - b\sum_{i=1}^{n} Y_i d(p_0, p_i). \tag{42}$$

Using (42)

$$F(p_0)) = F^*(p_0) + a\sum_{i=1}^{n} X_i d(p_0, p_i) + b\sum_{i=1}^{n} Y_i d(p_0, p_i). \tag{43}$$

By (41) and (43):

$$\begin{aligned} F^*(p_0^*) - F(p_0) &= F(p_0^*) - F^*(p_0) \\ &\quad -a\sum_{i=1}^{n}[d(p_0^*, p_i) + d(p_0, p_i)]X_i - b\sum_{i=1}^{n}[d(p_0^*, p_i) + d(p_0, p_i)]Y_i, \\ &= F(p_0^*) - F^*(p_0) - \sum_{i=1}^{n}(aX_i + bY_i)(d(p_0^*, p_i) + d(p_0, p_i)). \end{aligned}$$

Then

$$|F^*(p_0^*) - F(p_0)| \leq |F(p_0^*) - F^*(p_0)| + \sum_{i=1}^{n}(|a||X_i| + |b||Y_i|)(d(p_0^*, p_i) + d(p_0, p_i)).$$

Now since p_0 and $p_0^* \in D$, then $d(p_0, p_i) \leq diam(D)$ and $d(p_0^*, p_i) \leq diam(D)$, we have

$$|F^*(p_0^*) - F(p_0)| \leq |F(p_0^*) - F^*(p_0)| + 2diam(D)\big(|a|S_x + |b|S_y\big), \quad \forall p \in D,$$

where $S_x = \sum_{i=1}^{n} |X_i|$ y $S_y = \sum_{i=1}^{n} |Y_i|$ $\square$

5. Weber's Inverse Problem on the Sphere

In this section we extend Weber's inverse problem to the unit sphere. Let $n+1$ points $X_i = X(\phi_i, \theta_i) \in D\big(p_0, \frac{\pi}{4}\big) \subset S^2$, and positive weights $w_i \in \mathbb{R}$, $\forall i = 1, \ldots, n$.

The weights w_i of the points X_i must be modified to obtain new weights $w_i^* > 0$, so that the point X_0 is the weighted geometric median.

From (6), Weber's function is

$$F(\phi, \theta) = \sum_{i=1}^{n} w_i d_{S^2}(X(\phi, \theta), X(\phi_i, \theta_i)).$$

Theorem 18. *Let* $p_0 \in S^2$, $F : D\big(p_0, \frac{\pi}{4}\big) \to \mathbb{R}$ *the Weber function. Then the gradient*

$$\nabla F(\phi,\theta) = \left(\sum_{i=1}^{n} \frac{w_i(\sin\phi\cos\phi_i\cos(\theta-\theta_i) - \cos\phi\sin\phi_i)}{\cos(\frac{\alpha_i}{2})\|X-X_i\|}, \sum_{i=1}^{n} \frac{w_i\cos\phi\cos\phi_i\sin(\theta-\theta_i)}{\cos(\frac{\alpha_i}{2})\|X-X_i\|}\right). \tag{44}$$

Proof. It has

$$\begin{aligned} X(\phi,\theta) &= (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi), \\ X(\phi_i,\theta_i) &= (\cos\phi_i\cos\theta_i, \cos\phi_i\sin\theta_i, \sin\phi_i). \end{aligned}$$

Then

$$\|X-X_i\| = \sqrt{2-2\sin\phi\sin\phi_i\cos(\theta-\theta_i)}. \tag{45}$$

Then, partially deriving with respect to ϕ and θ in (45), we obtain

$$\frac{\partial}{\partial\phi}\|X-X_i\| = \frac{\sin\phi\cos\phi_i\cos(\theta-\theta_i) - \cos\phi\sin\phi_i}{\|X-X_i\|}, \tag{46}$$

$$\frac{\partial}{\partial\theta}\|X-X_i\| = \frac{\cos\phi\cos\phi_i\sin(\theta-\theta_i)}{\|X-X_i\|}. \tag{47}$$

By (8), we have

$$\sin\left(\frac{\alpha_i}{2}\right) = \frac{\|X(\phi,\theta) - X(\phi_i,\theta_i)\|}{2}, \quad \forall i = 1,\ldots,n. \tag{48}$$

Partially deriving with respect to ϕ and θ in (48), we have

$$\frac{\partial\alpha_i(\phi,\theta)}{\partial\phi} = \frac{1}{\cos\left(\frac{\alpha_i}{2}\right)}\frac{\partial}{\partial\phi}\|X-X_i\|. \tag{49}$$

By (46), we have

$$\frac{\partial\alpha_i(\phi,\theta)}{\partial\phi} = \frac{1}{\cos\left(\frac{\alpha_i}{2}\right)}\left(\frac{\sin\phi\cos\phi_i\cos(\theta-\theta_i) - \cos\phi\sin\phi_i)}{\|X-X_i\|}\right). \tag{50}$$

Analogously, we have

$$\frac{\partial\alpha_i(\phi,\theta)}{\partial\theta} = \frac{1}{\cos\left(\frac{\alpha_i}{2}\right)}\left(\frac{\cos\phi\cos\phi_i\sin(\theta-\theta_i)}{\|X-X_i\|}\right). \tag{51}$$

By (50) and (51), we obtain $\nabla F(\phi,\theta)$.

$$\nabla F(\phi,\theta) = \left(\sum_{i=1}^{n} \frac{w_i(\sin\phi\cos\phi_i\cos(\theta-\theta_i) - \cos\phi\sin\phi_i)}{\cos(\frac{\alpha_i}{2})\|X-X_i\|}, \sum_{i=1}^{n} \frac{w_i\cos\phi\cos\phi_i\sin(\theta-\theta_i)}{\cos(\frac{\alpha_i}{2})\|X-X_i\|}\right).$$

□

Let $X_i = X(\phi_i,\theta_i) \in D\left(p_0, \frac{\pi}{4}\right)$, $p_0 \in S^2$, $\forall i = 0,1,\ldots,n$, weights $w_i > 0$, $\forall i = 1,\ldots,n$. If $X_0 \neq X_i$, $\forall i = 1,\ldots,n$, for (3), the resultant force $R(X_0)$ at X_0 is given by

$$R(\phi,\theta) = -\nabla F(\phi,\theta) \tag{52}$$

Then

$$R(\phi,\theta) = \left(\sum_{i=1}^{n} w_i C_i, \sum_{i=1}^{n} w_i D_i\right), \tag{53}$$

where

$$\begin{aligned}
C_i &= \frac{-\sin\phi\cos\phi_i\cos(\theta-\theta_i)+\cos\phi\sin\phi_i}{\sin\alpha_i},\\
D_i &= \frac{-\cos\phi\cos\phi_i\sin(\theta-\theta_i))}{\sin\alpha_i}.
\end{aligned}$$

Theorem 19. *Let $X_i = X(\phi_i, \theta_i) \in D\big(p_0, \frac{\pi}{4}\big), p_0 \in S^2$, $\forall i = 1, \ldots, n$, weights $w_i > 0$, $\forall i = 1, \ldots, n$, $X_0^* = X(\phi_0^*, \theta_0^*) \in D\big(p_0, \frac{\pi}{4}\big)$, $X_0^* \neq X_i$, $\forall i = 1, \ldots, n$.*

If $w_i > \langle w, R_i\rangle$, $\quad \forall i = 1, \ldots, n$. Then Weber's inverse problem on the Sphere has a solution given by

$$w_i^* = w_i - aC_i - bD_i, \quad \forall i = 1, \ldots, n, \tag{54}$$

where

$$R_i = \frac{[C<D,D> - D<C,D>]C_i + [D<C,C> - C<C,D>]D_i}{\triangle}, \quad \forall i = 1, \ldots, n.$$

$$\begin{aligned}
C_i &= \frac{-\sin\phi_0^*\cos\phi_i\cos(\theta_0^*-\theta_i)+\cos\phi_0^*\sin\phi_i}{\sin\alpha_i},\\
D_i &= \frac{-\cos\phi_0^*\cos\phi_i\sin(\theta_0^*-\theta_i)}{\sin\alpha_i},\\
a &= \langle w, \frac{C<D,D> - D<C,D>}{\triangle}\rangle,\\
b &= \langle w, \frac{D<C,C> - C<C,D>}{\triangle}\rangle,
\end{aligned}$$

and $\triangle = \|C\|^2\|D\|^2 - \langle C, D\rangle^2 \neq 0$.

Proof. Since X_0^* is the weighted geometric median, then the resultant force given by (53) is

$$R(\phi_0^*, \theta_0^*) = \left(\sum_{i=1}^{n} w_i C_i, \sum_{i=1}^{n} w_i D_i\right),$$

where

$$\begin{aligned}
C_i &= \frac{-\sin\phi_0^*\cos\phi_i\cos(\theta_0^*-\theta_i)+\cos\phi_0^*\sin\phi_i}{\sin\alpha_i},\\
D_i &= \frac{-\cos\phi_0^*\cos\phi_i\sin(\theta_0^*-\theta_i)}{\sin\alpha_i}.
\end{aligned}$$

Next, consider the vectors

$$\begin{aligned}
w &= (w_1, \ldots, w_n),\\
C &= (C_1, \ldots, C_n),\\
D &= (D_1, \ldots, D_n).
\end{aligned}$$

Moreover, as $w_i > \langle w, R_i\rangle$, $\forall i = 1, \ldots, n$, then $w_i^N = w_i - aC_i - bD_i > 0$, $\forall i = 1, \ldots, n$.

Let $w_i^* = w_i^N$, $\quad i = 1, \ldots, n$. It is verified that $\langle w^*, C\rangle = 0$ and $\langle w^*, D\rangle = 0$.

Therefore, w^* is a solution of Weber's inverse problem on the Sphere. □

Next, we present the following algorithms to find a solution to Weber's inverse problem in the plane and on the unit sphere.

6. Numerical Examples

In this section we present some examples of Weber's inverse problem in the plane and on the unit sphere.

Algorithms 1 and 2 were coded and executed in MATLAB R2015a, running on Windows OS. The examples were carried out with AMD A12-9720P RADEON R7, 12 Compute Core 4C+8G 2.70 GHz and 12 GB RAM.

Algorithm 1 Pseudocode for solving the inverse Weber problem in the plane

---

Input:
  Different and non-collinear points $p_i = (x_i, y_i) \in \mathbb{R}^2, \forall i = 1, \ldots, n,$
  Vector of positive weights $w = (w_1, \ldots, w_n),$
  Weighted geometric median point $p_0^* = (x_0^*, y_0^*).$
Ouput:
  New weights $w^* = (w_1^*, \ldots, w_n^*).$

Compute the vectors
  $X = (\frac{x_1 - x_0^*}{d_1}, \frac{x_2 - x_0^*}{d_2}, \ldots, \frac{x_n - x_0^*}{d_n}), \qquad Y = (\frac{y_1 - y_0^*}{d_1}, \frac{y_2 - y_0^*}{d_2}, \ldots, \frac{y_n - y_0^*}{d_n})$
Where $d_i = \sqrt{(x_i - x_0^*)^2 + (y_i - y_0^*)^2}, \quad \forall\, i = 1, \ldots, n.$
  $\Delta = \|X\|^2\|Y\|^2 - \langle X, Y\rangle^2,$
  $a = \langle w, \frac{X<Y,Y> - Y<X,Y>}{\triangle}\rangle$
  $b = \langle w, \frac{Y<X,X> - X<X,Y>}{\triangle}\rangle$
  $w_i^N = w_i - aX_i - bY_i, \forall i = 1, \ldots, n.$
if $w_i^N > 0\ \forall i = 1, \ldots, n$ **then**
  A solution to Weber's inverse problem in the plane is given by
  $w_i^* = w_i^N, \ \forall i = 1, \ldots, n$
end if
Stop

Algorithm 2 Pseudocode for solving the inverse Weber problem on unit sphere

---

Input:
  Different points $X_i = X(\phi_i, \theta_i) \in S^2, \forall i = 1, \ldots, n,$
  Vector of positive weights $w = (w_1, \ldots, w_n),$
  Weighted geometric median point $X_0^* = X(\phi_0^*, \theta_0^*) = (X_0^1, X_0^2, X_0^3).$
Ouput:
  New weights $w^* = (w_1^*, \ldots, w_n^*).$
Compute the vectors
  $C = (C_1, \ldots, C_n), \qquad D = (D_1, \ldots, D_n),$
Where
$\alpha_i = a\cos\big(X_0^1\cos(\phi_i)\cos(\theta_i) + X_0^2\cos(\phi_i)\sin(\theta_i) + X_0^3\sin(\phi_i).\big)$
  $C_i = \frac{-\sin\phi_0^*\cos\phi_i\cos(\theta_0^* - \theta_i) + \cos\phi_0^*\sin\phi_i}{\sin\alpha_i},$
  $D_i = \frac{-\cos\phi_0^*\cos\phi_i\sin(\theta_0^* - \theta_i)}{\sin\alpha_i}.$
Compute
  $\Delta = \|C\|^2\|D\|^2 - \langle C, D\rangle^2,$
  $a = \langle w, \frac{C<D,D> - D<C,D>}{\triangle}\rangle$
  $b = \langle w, \frac{D<C,C> - C<C,D>}{\triangle}\rangle$
  $w_i^N = w_i - aC_i - bD_i, \forall i = 1, \ldots, n.$
if $w_i^N > 0\ \forall i = 1, \ldots, n$ **then**
  A solution to Weber's inverse problem on the sphere is given by
  $w_i^* = w_i^N, \ \forall i = 1, \ldots, n.$
end if
Stop

Example 1. *Consider 10 points in the plane, whose coordinates and associated weights are shown in Table 1.*

Table 1. Coordinates of the points and associated weights.

i	w_i	x_i	y_i
1	3.5	−5	2
2	2.2	−3	6
3	1.9	−2	4
4	2.8	2	3
5	3.7	6	4
6	2.7	5	1
7	3.3	7	−1
8	2.9	4	−3
9	3.6	1	−4
10	2.3	−1	−1

For the inverse Weber problem, we consider the point $p_0^* = (x_0^*, y_0^*) = (3, 2)$ *as the weighted geometric median given a priori. Using Algorithm 1, we obtain:*

Vector X

$$[-1.0000 \quad -0.8321 \quad -0.9285 \quad -0.7071 \quad 0.8321 \quad 0.8944 \quad 0.8000 \quad 0.1961 \quad -0.3162 \quad -0.8000]$$

Vector Y

$$[0.0000 \quad 0.5547 \quad 0.3714 \quad 0.7071 \quad 0.5547 \quad -0.4472 \quad -0.6000 \quad -0.9806 \quad -0.9487 \quad -0.6000]$$

The value of the parameters

$$\triangle = 22.7754, \qquad a = -0.8367, \qquad b = -1.4423.$$

The components of the normal vector w^N*:*

$$w_i^N = w_i - aX_i - bY_i \quad , \forall\, i = 1, \ldots, 10.$$

Vctor w^N

$$[2.6633 \quad 2.3039 \quad 1.6588 \quad 3.2283 \quad 5.1962 \quad 2.8033 \quad 3.1039 \quad 1.6498 \quad 1.9671 \quad 0.7653]$$

Since $w_i^N > 0, \quad \forall\, i = 1, \ldots, 10$*, a solution for Weber's inverse problem is:*

$$w^* = w^N.$$

Now, using the new weights w^* *obtained and the sequence (2) given by Weiszfeld's algorithm (1937) [12], we obtain the weighted geometric median of Weber's classical or direct problem, whose coordinates are* $(3, 2)$.

Furthermore, using the data in Table 1, and the Weiszfeld sequence (2), the coordinates of the weighted geometric median for the Weber problem are $(2.257920, 0.868847)$.

The following Figure 2 shows the 10 fixed points together with the weighted geometric medians $(2.257920, 0.868847)$ *and* $(3, 2)$*, of the classical and inverse Weber problem, respectively.*

Analogously, the point $p_0^* = (x_0^*, y_0^*) = (-1, 3)$ *is considered as the weighted geometric median given a priori (Figure 3). Using the Algorithm 1, the new weights* w_i^* *obtained are:*

Vector w^*

$$[4.20025 \quad 3.86680 \quad 3.56352 \quad 1.75205 \quad 2.84708 \quad 1.29327 \quad 1.77924 \quad 1.22688 \quad 2.05769 \quad 0.99539]$$

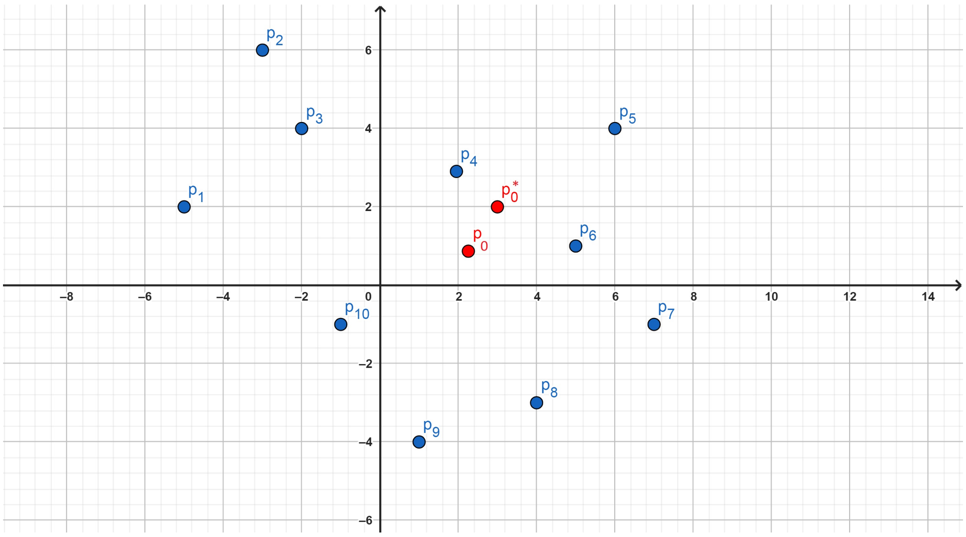

Figure 2. The point $(3,2)$ is the weighted geometric median for the inverse Weber problem.

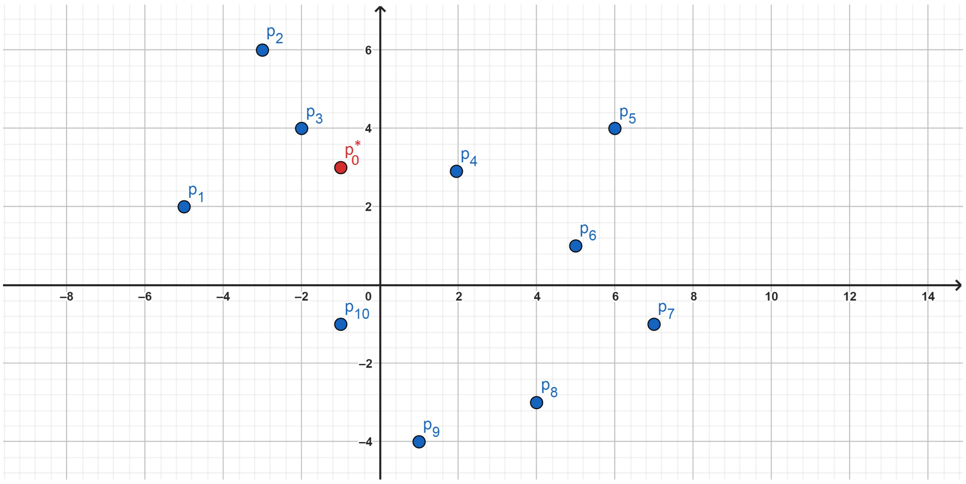

Figure 3. The point $(-1,3)$ is the weighted geometric median for Weber's inverse problem.

Example 2. *Now, we consider the example given by Burkard et al. (2010) [7] in their paper, whose points, weights, and bounds are given in Table 2.*

Table 2. Points, weights, and bounds in the example given by Burkard et al. [7]

i	w_i	$\underline{w}_i$	$\overline{w}_i$	$p_i(x_i, y_i)$
1	$\frac{50}{7}$	5	8	$p_1 = \left(-\frac{7}{25}, -\frac{24}{25}\right)$
2	$2\sqrt{2}$	1	3	$p_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
3	4	3	5	$p_3 = \left(\frac{3}{5}, \frac{4}{5}\right)$
4	3	3	4	$p_4 = \left(-\frac{4}{5}, \frac{3}{5}\right)$

In their paper, Burkard et al. [7] chose the point $p_0^* = (0,0)$ as the weighted geometric median, determining that the new weights w_i^* given by:

Vector w^*

$$\left[w_1^* = 5 \quad w_2^* = \frac{31}{35}\sqrt{2} \quad w_3^* = \frac{34}{7} \quad w_4^* = 3\right]$$

By choosing the point $p_0^* = (0,0)$ as the weighted geometric median given a priori, and using Algorithm 1, we obtain:

Vector X

$$\begin{bmatrix} -0.2800 & 0.7071 & 0.6000 & -0.8000 \end{bmatrix}$$

Vector Y

$$\begin{bmatrix} -0.9600 & -0.7071 & 0.8000 & 0.6000 \end{bmatrix}$$

The value of the parameters

$$\triangle = 3.7688, \qquad a = -0.2366, \qquad b = -1.6154.$$

The components of the normal vector w^N:

$$w_i^N = w_i - aX_i - bY_i, \quad \forall\, i = 1, 2, 3, 4.$$

$$\begin{bmatrix} w_1^N = 5.5258 & w_2^N = 1.8535 & w_3^N = 5.4343 & w_4^N = 3.7799 \end{bmatrix}$$

Since $w_i^N > 0, \quad \forall\, i = 1, 2, 3, 4$, a solution for Weber's inverse problem is:

$$w^* = w^N.$$

Note that the weights obtained by Burkard et al. and ours are different, which indicates that in this problem the solution is not unique. Figure 4 shows the fixed points and the weighted geometric median point.

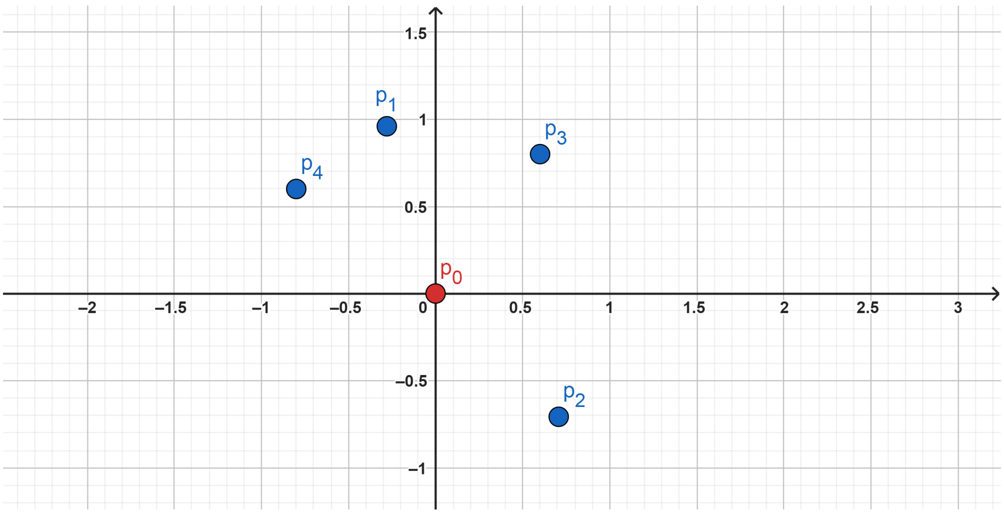

Figure 4. The point $(0, 0)$ is the weighted geometric median for Weber's inverse problem given by Burkard et al. [7].

Example 3. *Inspired by the example of Drezner, Z. and Wesolowsky, G.O.(1978) [16], in Service Location theory on the surface of the sphere, we consider the problem of distributing a product in 15 cities through air routes (Figure 5), whose information is given in Table 3.*

For the inverse Weber problem on the unit sphere, we a priori specify a city in which the Weber function must reach a minimum, necessitating the modification of the given weights w_i.

The following Figure 5 shows the distribution of cities over the surface of the unit sphere.

Table 4 shows four cities with their respective geographic coordinates in radians. Each of these cities will be considered as the weighted geometric median given a priori.

Using the data in Tables 3 and 4 we have the following:

Columns 5 and 6 of Table 3 contain the geographic coordinates in radians of the 15 cities, and using spherical coordinates the points on the unit sphere are obtained.

Column 7 contains the weights associated with these cities.

From Table 4, the geographical coordinates of the city of Milan in radians are $(\phi_0^*, \theta_0^*) = (0.79350, 0.1603872)$. Then, the point that will be the weighted geometric median given a priori is:

$$X_0^* = X(\phi_0^*, \theta_0^*) = X(0.79350, 0.1603872) = (0.6924, 0.1120, 0.7128) \in S^2,$$

where $X(\phi, \theta) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$ is the parameterization of the sphere given in spherical coordinates. Using Algorithm 2, we obtain the coordinates of the vectors C and D, Table 5:

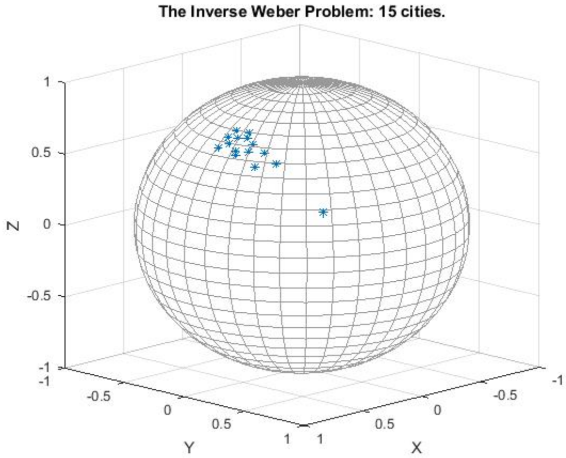

Figure 5. Distribution of cities on the surface of the sphere.

Table 3. Geographical coordinates of the cities and their respective initial weights.

I	Cities	Decimal Degree Latitude	Decimal Degree Longitude	Radian Laitude	Radian Longitude	Weights w_i
1	Paris	48.8534	2.3488	0.85265	0.040994	1.0
2	Amsterdam	52.3740	4.8896	0.91410	0.085340	1.0
3	Toulouse	43.60426	1.44367	0.76104	0.025197	1.0
4	Ginebra	46.20222	6.14569	0.806386	0.107263	1.0
5	Florence	43.78645	11.24892	0.76422	0.196331	1.0
6	Heidelberg	49.40768	8.69079	0.86233	0.151683	1.0
7	Rome	41.89193	12.51133	0.73115	0.218364	1.0
8	Berlin	52.52437	13.41053	0.91672	0.234058	1.0
9	Athens	37.98376	23.72784	0.66294	0.414129	1.0
10	Ankara	39.91987	32.85427	0.69673	0.573415	1.0
11	Warsaw	21.01178	52.22977	0.36672	0.911581	1.0
12	Prague	50.08804	14.42076	0.87420	0.251690	1.0
13	Bucarest	44.43225	26.10626	0.77549	0.455640	1.0
14	Sarajevo	43.84864	18.35644	0.76530	0.320380	1.0
15	Budapest	47.49835	19.04045	0.82900	0.332319	1.0

Table 4. Geographical coordinates of the cities given in radians.

I	City	Latitude: ϕ_0^*	Longitude: θ_0^*
1	Milan	0.79350	0.1603872
2	Saarbrücken	0.86005	0.1216543
3	Bern	0.81940	0.1299823
4	Vienna	0.84140	0.2857467

Table 5. Coordinates of the vectors C and D.

i	C_i	D_i
1	0.623210	−0.548498
2	0.935825	−0.247203
3	−0.273447	−0.674624
4	0.346511	−0.657903
5	−0.744649	0.468123
6	0.996628	−0.057545
7	−0.818345	0.403088
8	0.940616	0.238091
9	−0.493282	0.610087
10	−0.162344	0.692051
11	−0.345873	0.658068
12	0.815793	0.405625
13	0.019325	0.701224
14	−0.185022	0.689245
15	0.345762	0.658097

The value of the parameters

$$\triangle = 24.442, \qquad a = 0.55258, \qquad b = 0.89646.$$

The components of the normal vector w^N

$$w_i^N = w_i - aC_i - bD_i, \quad \forall\, i = 1, \ldots, 15.$$

Vector w^N

$$\begin{bmatrix} 1.14734 & 0.70449 & 1.75588 & 1.39831 & 0.99182 & 0.50088 & 1.09084 & 0.26680 \\ 0.72565 & 0.46931 & 0.60119 & 0.18559 & 0.36070 & 0.48435 & 0.21898 \end{bmatrix}$$

Since $w_i^N > 0, \quad \forall\, i = 1, \ldots, 15$, *a solution for Weber's inverse problem is:*

$$w^* = w^N.$$

The same procedure applies to the other cities: Saarbrücken, Bern, and Vienna.

In Table 6, the results of applying Algorithm 2 to determine the new weights w_i^* *in Weber's inverse problem are presented.*

Figure 6 shows the 15 cities with red dots and the 4 cities that correspond to the points that will be the weighted geometric medians (blue dots). The city with number 1 corresponds to Milan, number 2 corresponds to Saarbrücken, number 3 to Bern, and number 4 to Vienna. In these cities, the Weber function with its respective weights reaches a minimum.

Table 6. New weights for the inverse Weber problem.

I/City	w_i^*—Milan	w_i^*—Saarbrücken	w_i^*—Bern	w_i^*—Vienna
1	1.14734	1.59345	1.52396	0.65506
2	0.70449	1.73660	1.13560	1.01267
3	1.75588	0.95590	1.53783	0.33283
4	1.39831	0.60002	1.51823	0.39293
5	0.99182	0.21625	0.60894	0.15474
6	0.50088	0.41788	0.76234	0.75257
7	1.09084	0.21651	0.64668	0.15620
8	0.26680	0.85426	0.62411	1.48199
9	0.72565	0.15925	0.47595	0.61784
10	0.46931	0.21788	0.38543	1.06044
11	0.60119	0.19635	0.41636	1.03481
12	0.18559	0.47391	0.46571	1.33973
13	0.36070	0.24256	0.36844	1.10895
14	0.48435	0.16779	0.40806	0.46715
15	0.21898	0.27864	0.36446	1.18350

Figure 6. Cities: 1—Milan, 2—Saarbrücken, 3—Bern, and 4—Vienna, are the weighted geometric medians. (Source: Figure created by the authors based on a map obtained from Google Maps).

Example 4. *In this example, new weights are considered for the inverse Weber problem, (Table 7).*

Table 8 shows four cities with their respective geographic coordinates. Each of these cities will be considered as the weighted geometric median given a priori.

Table 9 shows the results of the application of our method to determine the new weights w_i^* in Weber's inverse problem.

Table 7. Geographical coordinates of the cities and new weights.

I	Cities	Decimal Degree Latitude	Decimal Degree Longitude	Radian Latitude	Radian Longitude	Weights w_i
1	Paris	48.8534	2.3488	0.85265	0.040994	0.90
2	Amsterdam	52.3740	4.8896	0.91410	0.085340	0.89
3	Toulouse	43.60426	1.44367	0.76104	0.025197	1.14
4	Ginebra	46.20222	6.14569	0.80638	0.107263	1.37
5	Florence	43.78645	11.24892	0.76422	0.196331	0.95
6	Heidelberg	49.40768	8.69079	0.86233	0.151683	0.86
7	Rome	41.89193	12.51133	0.73115	0.218364	0.74
8	Berlin	52.52437	13.41053	0.91672	0.234058	1.30
9	Athens	37.98376	23.72784	0.66294	0.414129	1.21
10	Ankara	39.91987	32.85427	0.69673	0.573415	0.93
11	Warsaw	21.01178	52.22977	0.36672	0.911581	1.52
12	Prague	50.08804	14.42076	0.87420	0.251690	1.13
13	Bucharest	44.43225	26.10626	0.77549	0.455640	0.92
14	Sarajevo	43.84864	18.35644	0.76530	0.320380	0.83
15	Budapest	47.49835	19.04045	0.82900	0.332319	1.00

Table 8. Geographical coordinates of the cities given in radians.

i	City	Latitude: ϕ_0^*	Longitude: θ_0^*
1	Milan	0.79350	0.1603872
2	Munich	0.84016	0.2020304
3	Bern	0.81940	0.1299823
4	Venice	0.79303	0.2152453

Table 9. New weights for the inverse Weber problem.

í / City	w_i^*—Milan	w_i^*—Munich	w_i^*—Bern	w_i^*—Venice
1	1.0314	1.1528	1.4529	0.9761
2	0.5654	1.2425	1.0687	0.7802
3	1.9111	1.1784	1.6594	1.4463
4	1.7627	1.4290	1.8677	1.5624
5	0.9625	0.6710	0.5217	1.3297
6	0.3280	1.1942	0.6601	0.7374
7	0.8546	0.4285	0.3479	1.0262
8	0.5331	1.5312	0.9556	0.9809
9	0.9466	0.8601	0.6570	1.1816
10	0.3985	0.6363	0.2987	0.7413
11	1.1268	1.2104	0.9141	1.3976
12	0.2846	1.2051	0.6145	0.7731
13	0.2737	0.6473	0.2772	0.6638
14	0.3143	0.4865	0.2170	0.6743
15	0.2014	0.7729	0.3624	0.6341

Figure 7 shows the 15 cities with red dots and the 4 cities that correspond to the points that will be the weighted geometric medians (blue dots). The city with number 1 corresponds to Milan, with number 2 corresponds to Munich, with number 3 to Bern, and with number 4 to Venice. In these cities, the Weber function with their respective weights reaches a minimum.

Figure 7. Cities: 1—Milan, 2—Munich, 3—Bern, and 4—Venecia, are the weighted geometric median. (Source: Figure created by the authors based on a map obtained from Google Maps.)

7. Conclusions

In this study, we present a new method based on the concepts of orthogonality to solve the inverse Weber problem in the plane and on the sphere.

In this paper we assume that the weighted geometric median given a priori is different from the fixed points, and that it lies inside the convex capsule of points.

Using the fixed points and the weighted geometric median given a priori, we obtain the vectors X and Y in the plane: Equations (12) and (13); and the vectors C and D in the case of the unit sphere (Theorem 19); and we conclude that the solution to Weber's inverse problem is found in the vector subspace orthogonal to the vectors X and Y in the case of the plane, and to the vectors C and D in the case of the unit sphere.

If the initial weights $w_i > <w, R_i>, \quad \forall\, i = 1, \ldots, n$, then from Theorems 13 and 19, it is concluded that Weber's inverse problem has a solution.

Moreover, if the initial weights verify $w_i > <w, R_i>, \forall\, i = 1, \ldots, n$, from Theorem 12, we conclude the existence of a lower bound for the minimum of the Weber function.

Another interesting result of our research is the determination of an upper bound for the difference between the minima of the direct and inverse Weber problems (40).

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