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On System of Variable Order Nonlinear p-Laplacian Fractional Differential Equations with Biological Application

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Abstract: The study of variable order differential equations is important in science and engineering for a better representation and analysis of dynamical problems. In the literature, there are several fractional order operators involving variable orders. In this article, we construct a nonlinear variable order fractional differential system with a p-Laplacian operator. The presumed problem is a general class of the nonlinear equations of variable orders in the ABC sense of derivatives in combination with Caputo's fractional derivative. We investigate the existence of solutions and the Hyers–Ulam stability of the considered equation. The presumed problem is a hybrid in nature and has a lot of applications. We have given its particular example as a waterborne disease model of variable order which is analysed for the numerical computations for different variable orders. The results obtained for the variable orders have an advantage over the constant orders in that the variable order simulations present the fluctuation of the real dynamics throughout our observations of the simulations.

Keywords: variable order operators; p-Laplacian operator; Hyers–Ulam stability; existence and uniqueness of solutions; numerical scheme

MSC: 34D20; 47A35; 60H10

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1. Introduction and Preliminaries

Many scientific and engineering studies have focused on the mathematical modeling and numerical simulation of dynamical systems. Adding more complexity to the classical models through using operators of fractional orders is a tried and true strategy. The development of fractional order operators includes both local and non-local kernels, as well as singular and non-singular kernels are of great interest in the community of researchers. There have been some fascinating new studies examining these factors. Experts have looked at broad categories of FDEs, including sequential FDEs, hybrid FDEs, and mixed FDEs, and many others have been studied in well-known works [1–5].

Analytical modeling, which uses a wide variety of mathematical techniques to characterize dynamic behaviors, benefits greatly from perturbation strategies. Despite the fact that, on occasion, it can be difficult to solve or evaluate a mathematical problem representing a particular dynamical system, by perturbing the system in certain way, it is feasible to utilize strategies for a wide range of findings.Dhage [6] has given a detailed explanation of the significance of the classification of hybrid differential equations in linear and quadratic perturbations using the first and second types. He also provided a historical context for several perturbation methods used in the theory of integral and differential formulations. Many different aspects of the solutions of second-order ordinary DEs with periodic boundaries and a special quadratic perturbation were investigated. Combining generalized Lipschitz and Caratheodory conditions, we prove the existence of extremal positive solutions and establish existence theorems for both the discontinuity and Caratheodory conditions. His findings confirmed some previously established results for periodic boundary value problems involving second-order ordinary differential equations. Zhao et al. [7] pioneered work on Riemann–Liouville differential operators in fractional hybrid differential equations. They proved an existence theorem for fractional hybrid differential equations under mixed Lipschitz and Carathéodory conditions. The existence of extremal solutions was also proved by establishing some fundamental fractional differential inequalities. The appropriate resources were thought about, and the comparison principle was proven, both of which are helpful in the quest of learning more about the subjective behavior of solutions. Abbas et al. [8] investigated a new family of fractional–integral hybrid DEs in which the derivatives of a function with respect to a particular continuously differentiable and increasing function. They produced a solution by using a hybrid fixed point (FP) theorem. They explored necessary conditions for the dependence to be continuous with respect to the specified parameters. A numerical example was presented at the end to show the validity of the results they achieved. Sutar and Kucche [9] formed the corresponding fractional integral equation and proved its existence using the nonlinear hybrid differential equations theory with fractional integrals including the Atangana–Baleanu–Caputo (ABC) fractional derivative. The uniqueness, existence of a maximal and minimal solution, and comparison results were also investigated, as was the development of the theory of inequalities for ABC-hybrid fractional differential equations.

The p-Laplacian operator is a nonlinear operator which was used in the modeling and qualitative aspects in a variety of problems. In mechanics, studying turbulence through a porous medium is fundamental. Leibenson [10] presented the p-Laplacian model for the research of such a problem given by $(\phi_p(u'))' = f(t, u, u')$ where $\phi_p(u) = |u|^{p-2}u$, where $p > 1$, and this operator has an inverse, which can be presented with ϕ_q. The p and q are interrelated with the following relationship: $1/p + 1/q = 1$ and $q > 1$. Khan et al. [11] considered an ABC-FDEs with a p-Laplacian operator incorporating a spatial singularity; we have studied the existence and uniqueness of these solutions as well as their Hyers–Ulam stability. A well-known Guo-Krasnoselskii theorem is used to derive the existence and uniqueness of solutions. Pang et al. [12] considered the case of a higher-order nonlinear differential equation with a p-Laplacian and a nonlocal boundary value problem at resonance. A few existence outcomes for the boundary value problems were acquired by employing a new extension theorem. In addition to the main findings, they also provided a concrete example to demonstrate those findings.

Bazighifan and Ragusa [13] considered a neutral nonlinear DE of fourth order driven by a p-Laplace operator, and discussed the oscillatory behavior of the solutions. The theory of analogy has yielded a few oscillation criteria for the understudy equation. As a result, the results obtained are an improvement on the previously published, but well-known, oscillation results. The results are shown to be useful in practice through an example. Devi et al. [14] considered hybrid p-Laplacian operator fractional differential equations (FDEs). Their primary focus was on analyzing the HU-stability of a class of hybrid FDEs including fractional derivatives of varying orders with the p-Laplacian operator and establishing the EU results for this class. Using the green function, they transformed the original problem into a hybrid form of FDEs applicable to European Union (EU) outcomes. An FP theorem was used to look into the possibility of an existence solution (ES), and the Banach contraction mapping principle was applied to determine the solution's uniqueness. More relevant articles can be studied in the references [15–17].

In 1993, Samko and Ross [18] first proposed the idea of VO integral and differential, along with a few of its fundamental features. In [19], Lorenzo and Hartley compiled the findings on VO fractional operators and explored the different ways in which these operators can be described. After that, the VO-FDE models were further investigated for their potential additions and useful applications [20]. Interest has increased during the previous ten years in this topic of study. For more related works on VO FDEs and their applications, methods, techniques some others [21–23], we suggest the works [24–26]. For some research works in relation to the tuberculosis and some others, see [27–32].

In science and engineering, studying differential equations of varying orders is crucial for accurately representing and analyzing dynamical situations. Many fractional order operators of varying orders can be found in the literature. Using the p-Laplacian operator, we build a nonlinear variable-order fractional differential system in this article. The assumed issue is highly versatile because of its hybrid nature. As a specific example, we have analyzed a variable-order model of water-borne disease using numerical calculations of varying orders. The advantage of the results produced for the variable orders over the constant orders is that the simulations of the variable orders will be analysed in the computational section of the paper. Our presumed problem is given by:

$$\begin{aligned} &{}^{c}\mathbb{D}^{\vartheta_i(t)}\left[\Phi_p\left({}^{ABC}\mathbb{D}^{\varrho_i(t)}\left(\mathcal{Y}_i(t)-\sum_{i=1}^{n}\mathbb{G}_i(t,\mathcal{Y}_i(t))\right)\right)\right]=\lambda_i^*(t,\mathcal{Y}_i(t)),\ t\in \mathrm{I}=[0,1],\\ &\mathcal{Y}_i(0)=\nu_i,\ \ \mathbb{G}_i(t,\mathcal{Y}_i(t))|_{t=0}=0, \end{aligned} \tag{1}$$

where $0<\varrho_i(t)\leq 1$, $\nu_i\in\mathbb{R}$, the functions $\mathcal{Y}_i:\mathrm{I}\to\mathbb{R}$ are differential functions with $\mathcal{Y}_i'$ are continuous for $i=1,2,\ldots,n$, λ_i^*, $\mathbb{G}_i:\mathrm{I}\times\mathbb{R}\to\mathbb{R}$, are continuous and hold the Caratheodory assumptions. The ${}^{ABC}\mathbb{D}^{\vartheta_i(t)}$, are the ABC-fractional differential operators of variable order while ${}^{c}\mathbb{D}^{\varrho_i(t)}$ are variable order Caputo's fractional differential operators for $i=1,2,\ldots,n$. The $\Phi_p(\mathfrak{U})=|\mathfrak{U}|^{p-2}\mathfrak{U}$ is the p-Laplacian operator with $1<p<2$ and $1/p+1/q=1$. There are no studies in the aforementioned body of research that, to the best of the our knowledge, address general hybrid research problems of this kind. In this paper, we investigate the EUS as well as the HU-stability, and present an application for the dynamic systems problem. In recent years, scientists have been concentrating their efforts on modeling system dynamics with fractional order operators. Both singular and non-singular kernels have been the subject of extensive research in current history. It might be challenging to determine which operator is the best fit for a certain situation; nonetheless, researchers always continue to look out for novel applications of a variety of operators. We point the researchers in the direction of the study [33–36] for additional details. This paper extends the majority of the conclusions examined for the ABC operator yet such a system (1) has not been investigated for the specified ABC-operator. Additionally, a lot of dynamical problems will have this work as a foundation for existence, uniqueness, and numerical simulations. The suggested mathematical model is far more complicated and n-coupled than any previous effort in the relevant literature.

In this work, we describe the required and sufficient conditions for the *ABC*–FDE hybrid system solutions to the proposed problem (1). A numerical formulation using Lagrange's interpolation polynomials will also be built and applied, and its use with a dynamical system will be demonstrated. In terms of the presence of solutions, we investigate the uniqueness and HU-stability of the suggested system. The findings are compared to the expected outcomes in order to evaluate the scheme's applicability and validity.

Next, we will provide some fundamental ideas from the variable order ABC calculus that will be utilized in the subsequent sections of the article to discuss the findings.

Definition 1 ([37]). *For $0 < \varepsilon(t) < 1$, and $f \in C^1(0,T) = \{f \in C(I,\mathbb{R}) : f' \in C(I,\mathbb{R})$ where $I = (0,T),$ for $T \in \mathbb{R}\}$, the VO ABC- fractional derivative is defined by*

$$ {}^{ABC}D_0^{\varepsilon(t)} f(t) = \frac{B(\varepsilon(t))}{1-\varepsilon(t)} \int_0^t (t-s)^{\varepsilon(t)-1} E_{\varepsilon(t)}(-\mu_{\varepsilon(t)}(t-s)^{\varepsilon(t)}) f'(s) ds, $$

for $f'(t) = \frac{d}{dt} f(t)$.

Definition 2 ([37,38]). *For a variable function $\varepsilon(t)$, with $\varepsilon : \mathbb{R} \to (0,1)$ and assuming that $f \in L^1(0,T)$, then the VO-AB-fractional integral is of f is given by*

$$ {}^{AB}I_0^{\varepsilon(t)} f(t) = \frac{B(1-\varepsilon(t))}{B(\varepsilon(t))} f(t) + {}^{RL}I_0^{\varepsilon(t)} f(t). \tag{2} $$

Lemma 1 ([37–39]). *For $f' \in L^1(0,\infty)$, and $0 < \varepsilon(t) < 1$, we derive*

$$ {}^{AB}I_0^{\varepsilon(t)}\, {}^{ABC}D_0^{\varepsilon(t)} f(t) = f(t) - f(0). $$

Lemma 2. *Suppose that Φ_p be p-Laplacian operator. Then,*

(1) *for $1 < p \le 2$, $\mathfrak{U}_1, \mathfrak{U}_2 > 0$, and $|\mathfrak{U}_1|, \mathfrak{U}_2 \ge \eta > 0$, we have*

$$ |\Phi_p(\mathfrak{U}_1) - \Phi_p(\mathfrak{U}_2)| \le (p-1)\eta^{p-2} |\mathfrak{U}_1 - \mathfrak{U}_2|; $$

(2) *for $p > 2$, and $|\mathfrak{U}_1|, |\mathfrak{U}_2| \le \eta$, we get*

$$ |\Phi_p(\mathfrak{U}_1) - \Phi_p(\mathfrak{U}_2)| \le (p-1)\eta^{p-2} |\mathfrak{U}_1 - \mathfrak{U}_2|. $$

2. Existence Criteria

In this section, the proposed VO p-Laplacian system is converted into its equivalent integral form and the existence of solutions is studied.

Lemma 3. *The n-coupled system of hybrid ABC-FDEs (1) has the kind of solutions*

$$ \begin{aligned} \mathcal{Y}_i =& \nu_i + \sum_{i=1}^{n} \mathbb{G}_i(t, \mathcal{Y}_i(t)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))} \Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))} \int_0^t (t-s)^{\vartheta_i(t)} \lambda_i^*(s, \mathcal{Y}_i(s)) ds\Big) \\ &+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))} \int_0^t (t-s)^{\varrho_i(t)-1} \\ &\times \Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))} \int_0^s (s-r)^{\vartheta_i(t)} \lambda_i^*(r, \mathcal{Y}_i(r)) dr\Big) ds, \end{aligned} $$

for $i = 1, 2, \ldots, n$.

Proof. By utilizing $({}^{AB}I_t^{\vartheta_i})$ to the variable order Φ_p system (1), for $i = 1, 2, \ldots, n$, we get

$$ \Phi_p\Big({}^{ABC}D^{\varrho_i(t)}\Big(\mathcal{Y}_i(t) - \sum_{i=1}^{m} \mathbb{G}_i(t, \mathcal{Y}_i(t))\Big)\Big) = c_1 + I^{\vartheta_i(t)} \lambda_i^*(t, \mathcal{Y}_i(t)), $$

where $i = 1, 2, \ldots, n$. By $\mathcal{Y}_i(0) = \nu_i$, and $\mathbb{G}_i(t, \mathcal{Y}_i(t))|_0 = 0$, it follows that $c_1 = 0$. Thus, we derive

$$ \Phi_p\Big({}^{ABC}D^{\varrho_i(t)}\Big(\mathcal{Y}_i(t) - \sum_{i=1}^{m} \mathbb{G}_i(t, \mathcal{Y}_i(t))\Big)\Big) = I^{\vartheta_i(t)} \lambda_i^*(t, \mathcal{Y}_i(t)). $$

Using the p-Laplacian operator, we obtain

$$^{ABC}D^{\varrho_i(t)}\Big(\mathcal{Y}_i(t) - \sum_{i=1}^{m}\mathbb{G}_i(t,\mathcal{Y}_i(t))\Big) = \Phi_q\Big(I^{\vartheta_i(t)}\lambda_i^*(t,\mathcal{Y}_i(t))\Big).$$

This can also be expressed as the following formulation:

$$\mathcal{Y}_i(t) = \nu_i + \sum_{i=1}^{m}\mathbb{G}_i(t,\mathcal{Y}_i(t)) +^{AB} I^{\varrho_i(t)}\Phi_q\Big(I^{\vartheta_i(t)}\lambda_i^*(t,\mathcal{Y}_i(t))\Big)$$

or

$$\begin{aligned}\mathcal{Y}_i =& \nu_i + \sum_{i=1}^{n}\mathbb{G}_i(t,\mathcal{Y}_i(t)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s,\mathcal{Y}_i(s))ds\Big)\\ &+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\\ &\times \Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r,\mathcal{Y}_i(r))dr\Big)ds.\end{aligned}$$

The proof is completed. □

Consider a Banach's space $\mathbb{B} = \{\mathcal{Y}_i(t) : \mathcal{Y}_i(t) \in \mathbb{C}([0,1],\mathbb{R}) \text{ for } t \in [0,1]\}$, possessing the norm $\|\mathcal{Y}_i\| = \max_{t\in[0,1]}|\mathcal{Y}_i(t)|$, and $i = 1,2,\ldots,n$. Supposing $\mathbb{T}_i : \mathbb{C}([0,1],\mathbb{R}) \to \mathbb{C}([0,1],\mathbb{R})$, operators for $i = 1,2,\ldots,n$, where

$$\begin{aligned}\mathbb{T}_i\mathcal{Y}_i(t) =\; & \nu_i + \sum_{i=1}^{n}\mathbb{G}_i(t,\mathcal{Y}_i(t)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s,\mathcal{Y}_i(s))ds\Big)\\ &+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\\ &\times \Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r,\mathcal{Y}_i(r))dr\Big)ds. \end{aligned} \tag{3}$$

The system (3) implies that all the FPs of $\mathbb{T}_i$ are solutions of integral coupled system (1). The proofs of our findings are based on the above conditions:

- (Q_1) Let $a,\, b > 0$. For the functions λ_i $i = 1,2,3,\ldots,n$, we obtain

$$|\lambda_i(t,\mathcal{Y}_i(t))| \le \Phi_p(a + b\|\mathcal{Y}_i\|).$$

- (Q_2) Let $\nu_i^1, \nu_i^2 \in \mathbb{R}_e$, and $\mathcal{Y}_i,\, \bar{\mathcal{Y}}_i \in \mathbb{C}$, $t \in [0,k]$. We suppose that

$$|\lambda_i^*(t,\mathcal{Y}_i) - \lambda_i^*(t,\bar{\mathcal{Y}}_i)| \le \nu_i^1\|\mathcal{Y}_i - \bar{\mathcal{Y}}_i\|,$$

$$|\mathbb{G}_i(t,\mathcal{Y}_i) - \mathbb{G}_i(t,\bar{\mathcal{Y}}_i)| \le \nu_i^2\|\mathcal{Y}_i - \bar{\mathcal{Y}}_i\|.$$

Lemma 4. *Let the conditions (Q_1) and (Q_2) are satisfied and for*

$$\begin{aligned}\chi_i = \max_{t\in[0,1]}\Bigg\{&\sum_{i=1}^{n}\nu_i^2 + \frac{(1-\varrho_i(t))(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)}}{\Gamma(\vartheta_i(t)+1)}\\ &+ \frac{(\varrho_i(t))(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)+\varrho_i(t)}}{\Gamma(\vartheta_i(t)+1)}\Bigg\},\end{aligned} \tag{4}$$

we have $\chi_i < 1$, $i = 1,2,\ldots,n$. Hence, the variable order Φ_p-system (1) has a unique solution.

Proof. For $i = 1, 2, \ldots, n$, suppose that $\sup_{t\in[0,k]} |\mathbb{G}_i(t, 0)| = \wp_2 < \infty$, and $\sup_{t\in[0,k]} |\lambda_i^*(t, 0)| = \wp_1 < \infty$, $\mathbb{S}_{\eta_i} = \{\mathcal{Y}_i \in \mathbb{C}([0, k], \mathbb{R}_e) : \|\mathcal{Y}_i\| < \eta_i\}$, and $k \geq 1$. For $\mathcal{Y}_i \in \mathbb{S}_{\eta_i}$, and $t \in [0, k]$, then we proceed to:

$$\begin{aligned}
|\Phi_q\Big(I^{\vartheta_i(t)}\lambda_i^*(t,\mathcal{Y}_i(t))\Big)| =& |\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)-1}\lambda_i^*(s,\mathcal{Y}_i(s))ds\Big)| \\
\leq& \Big|\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)-1}\Phi_p(a+b\|\mathcal{Y}_i\|)\Big)\Big| \\
\leq& \Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t)+1)}(t-s)^{\vartheta_i(t)}\Big)(a+b\eta_i) \\
=& \frac{(a+b\eta_i)}{(\Gamma(\vartheta_i(t)+1))^{q-1}}T^{\vartheta_i(t)(q-1)}.
\end{aligned} \tag{5}$$

Also, for $\mathcal{Y}_i \in \mathbb{S}_{\eta_i}$, $t \in [0, k]$, the following is obtained:

$$\begin{aligned}
|\mathbb{G}_i(t,\mathcal{Y}_i(t))| =& |\mathbb{G}_i(t,\mathcal{Y}_i(t)) - \mathbb{G}_i(t,0) + \mathbb{G}_i(t,0)| \\
\leq& |\mathbb{G}_i(t,\mathcal{Y}_i(t)) - \mathbb{G}_i(t,0)| + |\mathbb{G}_i(t,0)| \\
\leq& \nu_i^2|\mathcal{Y}_i(t)| + |\mathbb{G}_i(t,0)| \\
\leq& \nu_i^2\eta_i + \wp_2.
\end{aligned} \tag{6}$$

By (3), (5) and (6), we obtain

$$\begin{aligned}
|\mathbb{T}_i\mathcal{Y}_i(t)| =& \Big|\nu_i + \sum_{i=1}^n \mathbb{G}_i(t,\mathcal{Y}_i(t)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s,\mathcal{Y}_i(s))ds\Big) \\
&+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r,\mathcal{Y}_i(r))dr\Big)ds\Big| \\
\leq& \nu_i + n(\nu_i^2\eta_i + \wp_2) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\frac{(a+b\eta_i)}{(\Gamma(\vartheta_i(t)+1))^{q-1}}T^{\vartheta_i(t)(q-1)} \\
&+ \frac{(a+b\eta_i)}{\Gamma(\vartheta_i(t)+1)}\frac{\varrho_i(t)(T^{\varrho_i(t)+\vartheta_i(t)})^{(q-1)}}{B(\varrho_i(t))\Gamma(\varrho_i(t)+1)}.
\end{aligned}$$

Thus, $\mathbb{T}_i\mathbb{S}_{\eta_i} \subset \mathbb{S}_{\eta_i}$. Next, let $\mathcal{Y}_i, v_i \in \mathbb{C}([0, 1], \mathbb{R}_e)$, $k \geq 1$ for $t \geq s \in [0, 1]$. Then, we obtain

$$\begin{aligned}
|\mathbb{T}_i\mathcal{Y}_i(t) - \mathbb{T}_i v_i(t)| =& \Big|\nu_i + \sum_{i=1}^n \mathbb{G}_i(t,\mathcal{Y}_i(t)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))} \\
&\times \Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s,\mathcal{Y}_i(s))ds\Big) \\
&+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r,\mathcal{Y}_i(r))dr\Big)ds \\
&- \Big[\nu_i + \sum_{i=1}^n \mathbb{G}_i(t,v_i(t)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s,v_i(s))ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r,v_i(r))dr\Big)ds\Big]\Big|\\
&\leq \sum_{i=1}^n \nu_i^2\|\mathcal{Y}_i-v_i\| + \frac{(1-\varrho_i(t))(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)}}{\Gamma(\vartheta_i(t)+1)}\|\mathcal{Y}_i-v_i\|\\
&+\frac{\varrho_i(t)(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)+\varrho_i(t)}}{\Gamma(\vartheta_i(t)+1)}\|\mathcal{Y}_i-v_i\|\\
&\leq \Big(\sum_{i=1}^n \nu_i^2 + \frac{(1-\varrho_i(t))(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)}}{\Gamma(\vartheta_i(t)+1)}\\
&+\frac{\varrho_i(t)(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)+\varrho_i(t)}}{\Gamma(\vartheta_i(t)+1)}\Big)\|\mathcal{Y}_i-v_i\|\\
&=\chi_i\|\mathcal{Y}_i-v_i\|.
\end{aligned}
$$

For $\chi_i < 1$, where χ_i's are defined by (4). Thus, $\mathbb{T}_i$ are contractions and Banach's FP theorem implies that the variable order Φ_p system (1) has solutions uniqueness, where $i = 1, 2, 3, \ldots, n$. □

Theorem 1. *Subjected to the conditions of Lemma 4, the variable order Φ_p-system (1) has a solution.*

Proof. By Lemma 4, the operators $\mathbb{T}_i$ are bounded and let for t_1, $t_2 \in [0, k]$ with $t_2 > t_1$, and $k \leq 1$, we have

$$
\begin{aligned}
|\mathbb{T}_i w(t_2) - \mathbb{T}_i w(t_1)| = \Big|&\nu_i + \sum_{i=1}^n \mathbb{G}_i(t_2,\mathcal{Y}_i(t_2))\\
&+\frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^{t_2}(t_2-s)^{\vartheta_i(t)}\lambda_i^*(s,\mathcal{Y}_i(s))ds\Big)\\
&+\frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^{t_2}(t_2-s)^{\varrho_i(t)-1}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r,\mathcal{Y}_i(r))dr\Big)ds\\
&-\Big[\nu_i+\sum_{i=1}^n \mathbb{G}_i(t_1,\mathcal{Y}_i(t_1)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^{t_1}(t_1-s)^{\vartheta_i(t)}\lambda_i^*(s,\mathcal{Y}_i(s))ds\Big)\\
&+\frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^{t_1}(t_1-s)^{\varrho_i(t)-1}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r,\mathcal{Y}_i(r))dr\Big)ds\Big)\Big]\Big|\\
&\leq \sum_{i=1}^n \big|\mathbb{G}_i(t_2,\mathcal{Y}_i(t_2)) - \mathbb{G}_i(t_1,\mathcal{Y}_i(t_1))\big|\\
&+\frac{1-\varrho_i(t)}{B(\varrho_i(t))}\frac{(q-1)\rho^{q-2}(a_i+b_i\eta_i)}{\Gamma(\vartheta_i(t)+1)}|t_2^{\vartheta_i(t)} - t_1^{\vartheta_i(t)}|\\
&+\frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t)+1)}\frac{(q-1)}{}\frac{(a+b\eta_i)}{(\Gamma(\vartheta_i(t)+1))^{q-1}}T^{\vartheta_i(t)(q-1)}|t_2^{\vartheta_i(t)} - t_1^{\vartheta_i(t)}||.
\end{aligned}
\tag{7}
$$

Since $\mathbb{G}_i$, $(i = 1, 2, 3, \ldots, n)$, are continuous functions, then it follows from (7) that, as $t_2 \to t_1$, we have $\mathbb{T}_i w(t_2) \to \mathbb{T}_i w(t_1)$. Equivalently, $\mathbb{T}_i$ are equicontinuous operators. Additionally, for $u \in \{u \in \mathbb{C}([0,k], \mathbb{R}_e) : u = \hbar\mathbb{T}_i(u),\ for \hbar \in [0,1]\}$, we obtain

$$
\begin{aligned}
\|\mathcal{Y}_i\| =& \max_{t\in I} |\mathbb{T}_i\mathcal{Y}_i(t)| = \Big|\nu_i + \sum_{i=1}^{n} \mathbb{G}_i(t,\mathcal{Y}_i(t)) \\
&+ \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s,\mathcal{Y}_i(s))ds\Big) \\
&+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r,\mathcal{Y}_i(r))dr\Big)ds\Big| \\
\leq& \nu_i + n\wp_2 + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\frac{a}{(\Gamma(\vartheta_i(t)+1))^{q-1}}T^{\vartheta_i(t)} \\
&+ \frac{a}{(\Gamma(\vartheta_i(t)+1))^{q-1}}\frac{\varrho_i(t)T^{\varrho_i(t)+\vartheta_i(t)}}{B(\varrho_i(t))\Gamma(\varrho_i(t)+1)} \\
&+ \Big[n\nu_i^2 + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\frac{b}{(\Gamma(\vartheta_i(t)+1))^{q-1}}T^{\vartheta_i(t)} \\
&+ \frac{b}{(\Gamma(\vartheta_i(t)+1))^{q-1}}\frac{\varrho_i(t)T^{\varrho_i(t)+\vartheta_i(t)}}{B(\varrho_i(t))\Gamma(\varrho_i(t)+1)}\Big]\|\mathcal{Y}_i\| \\
=& \Delta^*_{i\,1} + \Delta^*_{i\,2}\|\mathcal{Y}_i\|.
\end{aligned} \tag{8}
$$

For $i = 1,2,3,\dots,n$,

$$
\begin{aligned}
\Delta^*_{i\,1} =& \nu_i + n\wp_2 + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\frac{a}{(\Gamma(\vartheta_i(t)+1))^{q-1}}T^{\vartheta_i(t)} \\
&+ \frac{a}{(\Gamma(\vartheta_i(t)+1))^{q-1}}\frac{\varrho_i(t)T^{\varrho_i(t)+\vartheta_i(t)}}{B(\varrho_i(t))\Gamma(\varrho_i(t)+1)},
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\Delta^*_{i\,2} =& n\nu_i^2 + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\frac{b}{(\Gamma(\vartheta_i(t)+1))^{q-1}}T^{\vartheta_i(t)} \\
&+ \frac{b}{(\Gamma(\vartheta_i(t)+1))^{q-1}}\frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t)+1)}T^{\varrho_i(t)+\vartheta_i(t)},
\end{aligned} \tag{10}
$$

for $i = 1,2,\dots,n$. From (8), (9) and (10), we deduce that

$$
\|\mathcal{Y}_i\| \leq \frac{\Delta^*_{i\,1}}{1-\Delta^*_{i\,2}},
$$

for $i = 1,2,\dots,n$. As a result, the Leray–Schauder alternative theorem is met, and (1) has a solution. □

3. Hyers–Ulam Stability

The research on the HU-stability of the n-coupled system is undertaken in this section with the aid of certain helpful articles [40–42].

Definition 3. *The coupled system of integral Equation (3) is HU-stable if for some $\nu_i > 0$, we have $\Delta_i > 0$, with $\mathcal{Y}_i$ so that*

$$
\|\mathcal{Y}_i - \mathbb{T}_i\mathcal{Y}_i\|_1 < \Delta_i, \tag{11}
$$

with $\bar{\mathcal{Y}}_i(t)$ of the coupled-system (3) with

$$
\bar{\mathcal{Y}}_i(t) = \mathbb{T}_i\bar{\mathcal{Y}}_i(t), \tag{12}
$$

and

$$\|\mathcal{Y}_i - \bar{\mathcal{Y}}_i\| < \Delta_i \nu_i,$$

where $i = 1, 2, \dots, n$.

Theorem 2. *Suppose the conditions of Lemma 4 are satisfied. Then, (3) is HU-stable; equivalently, the n-coupled hybrid system of ABC-FDEs (1) is HU-stable.*

Proof. Suppose for $\mathcal{Y}_i \in \mathbb{C}$ with the feature (11) and let $w_i^* \in \mathbb{C}$ for the coupled-system (1) fulfilling (3). Then,

$$\begin{aligned}
|\mathbb{T}_i\mathcal{Y}_i(t) - \mathbb{T}_i w_i^*(t)| = \Big| & \nu_i + \sum_{i=1}^{n} \mathbb{G}_i(t, \mathcal{Y}_i(t)) \\
& + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s, \mathcal{Y}_i(s))ds\Big) \\
& + \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r, \mathcal{Y}_i(r))dr\Big)ds \\
& - \Big[\nu_i + \sum_{i=1}^{n} \mathbb{G}_i(t, w_i^*(t)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s, w_i^*(s))ds\Big) \\
& + \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r, w_i^*(r))dr\Big)ds\Big]\Big|
\end{aligned}$$

$$\begin{aligned}
& \le \sum_{i=1}^{n} \nu_i^2|\mathcal{Y}_i - w_i^*| + \frac{(1-\varrho_i(t))(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)}}{\Gamma(\vartheta_i(t)+1)}|\mathcal{Y}_i - w_i^*| \\
& + \frac{\varrho_i(t)(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)+\varrho_i(t)}}{\Gamma(\vartheta_i(t)+1)}|\mathcal{Y}_i - w_i^*| \\
& \le \Big(\sum_{i=1}^{n} \nu_i^2 + \frac{(1-\varrho_i(t))(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)}}{\Gamma(\vartheta_i(t)+1)} \\
& + \frac{\varrho_i(t)(p-1)\rho^{q-2}}{B(\varrho_i(t))}\nu_i^1\frac{T^{\vartheta_i(t)+\varrho_i(t)}}{\Gamma(\vartheta_i(t)+1)}\Big)|\mathcal{Y}_i - w_i^*| \\
& = \chi_i|\mathcal{Y}_i - w_i^*|.
\end{aligned} \tag{13}$$

For $\chi_i < 1$, where χ_is are provided by (4), for $i = 1, 2, \dots, n$. Using the (11), (12) and (13), we take into

$$\begin{aligned}
\|\mathcal{Y}_i - \varpi_i^*\| &= \|\mathcal{Y}_i - \mathbb{T}_i\mathcal{Y}_i + \mathbb{T}_i\mathcal{Y}_i - \varpi_i^*\| \\
&\le \|\mathcal{Y}_i - \mathbb{T}_i\mathcal{Y}_i\| + \|\mathbb{T}_i\mathcal{Y}_i - \mathbb{T}_i\varpi_i^*\| \\
&\le \Delta_i + \chi_i\|\mathcal{Y}_i - \varpi_i^*\|,
\end{aligned}$$

where $i = 1, 2, \dots, m$. Additionally,

$$\|\mathcal{Y}_i - \varpi_i^*\| \le \frac{\Delta_i}{1-\chi_i},$$

with $\nu_i = \frac{1}{1-\chi_i}$. The linked system (3) is therefore stable. This further suggests that the coupled Hybrid ABC-FDEs system, (1), is stable. □

4. Computational Scheme for the *ABC*-FDEs

In this section, we construct a numerical approach for a broad category of p-Laplacian-operated ABC-FDEs as described in (1). We take consideration the nonlinear system provided in (3), with $\mathcal{Y}_i$, for $i = 1, 2, \ldots, n$, which leads to the following:

$$\begin{aligned}\mathcal{Y}_i(t) =& \nu_i + \sum_{i=1}^{n} \mathbb{G}_i(t, \mathcal{Y}_i(t)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s, \mathcal{Y}_i(s))ds\Big) \\ &+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^s (s-r)^{\vartheta_i(t)}\lambda_i^*(r, \mathcal{Y}_i(r))dr\Big)ds. \end{aligned} \tag{14}$$

Because the p-Laplacian is a nonlinear operator, nonlinear functions can be defined for $i = 1, 2, \ldots, n$:

$$\mathbb{B}_i(t, \mathcal{Y}_i(t)) = \Phi_q\Big(\frac{1}{\Gamma(\vartheta_i(t))}\int_0^t (t-s)^{\vartheta_i(t)}\lambda_i^*(s, \mathcal{Y}_i(s))ds\Big),$$

where $\mathbb{B}_i(t, \mathcal{Y}_i(t))|_0 = 0$. Then, (14) appears as follows:

$$\begin{aligned}\mathcal{Y}_i(t) =& \nu_i + \sum_{i=1}^{n} \mathbb{G}_i(t, \mathcal{Y}_i(t)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\mathbb{B}_i(t, \mathcal{Y}_i(t)) \\ &+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^t (t-s)^{\varrho_i(t)-1}\mathbb{B}_i(s, \mathcal{Y}_i(s))ds.\end{aligned}$$

Using Lagrange's interpolation polynomials, for this system, we develop a numerical scheme. Substituting t with t_{n+1}, we obtain

$$\begin{aligned}\mathcal{Y}_i(t_{n+1}) =& \nu_i + \sum_{i=1}^{n} \mathbb{G}_i(t_n, \mathcal{Y}_i(t_n)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\mathbb{B}_i(t_n, \mathcal{Y}_i(t_n)) \\ &+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\int_0^{t_{n+1}} (t_{n+1}-s)^{\varrho_i(t)-1}\mathbb{B}_i(s, \mathcal{Y}_i(s))ds.\end{aligned} \tag{15}$$

Using Lagrange's interpolation, we obtain

$$\begin{aligned}\mathbb{B}_i(t, \mathcal{Y}_i(t)) =& \frac{\mathbb{B}_i(t_k, \mathcal{Y}_i(t_k)(t-t_{k-1})}{t_k - t_{k-1}} - \frac{\mathbb{B}_i(t_{k-1}, \mathcal{Y}_i(t_{k-1}))(t-t_k)}{t_k - t_{k-1}} \\ =& \frac{\mathbb{B}_i(t_k, \mathcal{Y}_i(t_k))(t-t_{k-1})}{h} - \frac{\mathbb{B}_i(t_{k-1}, \mathcal{Y}_i(t_{k-1})(t-t_k)}{h}.\end{aligned} \tag{16}$$

With the help of (15) and (16), we have

$$\begin{aligned}\mathcal{Y}_i(t_{k+1}) =& \nu_i + \sum_{i=1}^{n} \mathbb{G}_i(t_k, \mathcal{Y}_i(t_k)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\mathbb{B}_i(t_k, \mathcal{Y}_i(t_k)) \\ &+ \frac{\varrho_i(t)}{B(\varrho_i(t))\Gamma(\varrho_i(t))}\Sigma_{i=1}^{n}\Big[\frac{\mathbb{B}_i(t_i, u(t_i))}{h}\int_{t_k}^{t_{k+1}}(\nu - t_{i-1})(t_{n+1}-\nu)^{\varrho_1 - 1}d\nu \\ &- \frac{\mathbb{B}_i(t_{i-1}, \mathcal{Y}_i(t_{i-1}))}{h}\int_{t_k}^{t_{n+1}}(\nu - t_i)(t_{n+1}-\nu)^{\varrho_1 - 1}d\nu\Big].\end{aligned}$$

Integration produces the following:

$$\begin{aligned}\mathcal{Y}_{k+1} =& \nu_i + \sum_{j=1}^{k} \mathbb{G}_i(t_k, \mathcal{Y}_i(t_k)) + \frac{1-\varrho_i(t)}{B(\varrho_i(t))}\mathbb{B}_i(t_k, \mathcal{Y}_i(t_k)) \\ &+ \frac{\varrho_1 h^{\varrho_1}}{\Gamma(\varrho_1+2)}\Sigma_{j=1}^{k}\Bigg[\mathbb{B}_i(t_i, \mathcal{Y}_i(t_i))\Bigg((k-j+1)^{\varrho_1}(k+2-j+\varrho_1) \\ &- (k-i)^{\varrho_1}(k+2-j+2\varrho_1)\Bigg) \\ &- \mathbb{B}_i(t_{i-1}, \mathcal{Y}_{i-1})\Bigg((k-i+1)^{\varrho_1+1} - (k-j+1+\varrho_1)(k-i)^{\varrho_1}\Bigg)\Bigg].\end{aligned}$$

The works [43–46] contains more beneficial numerical techniques.

Modeling a Waterborne Disease Using the Numerical Method

Diseases that are spread through contact with contaminated water are called waterborne diseases. These diseases can have devastating effects on human health, including death and disability. These infectious diseases can be spread through the use of contaminated water for drinking, washing, or cooking purposes. They are an acute issue in the countryside of developing nations everywhere. In addition to gastrointestinal symptoms, waterborne illnesses can cause problems with the skin, ears, respiratory system, and eyes. When a community does not have access to safe water, proper sanitation, and hygiene education, it is more likely to experience an outbreak of waterborne diseases. Therefore, the primary strategy for avoiding water-related illnesses is ensuring that people have access to safe drinking water and sanitation facilities. This section is based on [47]. We define a waterborne mathematical model with variable order as:

$$\begin{cases} {}_0^{ABC}\mathfrak{D}_t^{\alpha(t)}\dot{\mathcal{S}}(t) = \mu\mathfrak{N} - (1-r_1)b_w\dot{\mathcal{S}}\dot{\mathcal{W}} - (1-r_3)b_d\dot{\mathcal{S}}\dot{\mathfrak{D}} - b_l\dot{\mathcal{S}}\dot{\mathfrak{I}} - \mu\dot{\mathcal{S}}, \\ {}_0^{ABC}\mathfrak{D}_t^{\alpha(t)}\dot{\mathfrak{I}}(t) = (1-r_1)b_w\dot{\mathcal{S}}\dot{\mathcal{W}} + (1-r_3)b_d\dot{\mathcal{S}}\dot{\mathfrak{D}} + b_l\dot{\mathcal{S}}\dot{\mathfrak{I}} - \gamma\dot{\mathfrak{I}} - \mu\dot{\mathfrak{I}}, \\ {}_0^{ABC}\mathfrak{D}_t^{\alpha(t)}\dot{\mathcal{W}}(t) = (1-r_2)\alpha(t)\dot{\mathfrak{I}} + (1-r_3)\lambda\dot{\mathfrak{D}} - \xi_w\dot{\mathcal{W}}, \\ {}_0^{ABC}\mathfrak{D}_t^{\alpha(t)}\dot{\mathfrak{D}}(t) = (1-r_2)\alpha(t)_2\dot{\mathfrak{I}} - (1-r_3)\lambda\dot{\mathfrak{D}} - \xi_d\dot{\mathfrak{D}}, \\ {}_0^{ABC}\mathfrak{D}_t^{\alpha(t)}\dot{\mathfrak{R}}(t) = \gamma\dot{\mathfrak{I}} - \mu\dot{\mathfrak{R}}. \end{cases} \tag{17}$$

The entire populace $\mathfrak{N}$ is split into: $\dot{\mathcal{S}}(t), \dot{\mathfrak{I}}(t), \dot{\mathcal{W}}(t), \dot{\mathfrak{D}}(t)(t)$, and $\dot{\mathfrak{R}}(t)$ so that the density of sensitive individuals is $\dot{\mathcal{S}}$. The number of infected individuals that are actively spreading the disease is $\dot{\mathfrak{I}}$, and the amount of pathogensv within the water tank is $\dot{\mathcal{W}}$. The quantity of pathogens at the disposal site is $\dot{\mathfrak{D}}$, and the recovered individual density is $\dot{\mathfrak{R}}$.

The parameters are also as follows: b_l is the contact rate of persons, b_w represents the rate of contact for the reservoir-person, b_d is the contact rate of persons with the dumps, γ is infection time, ξ_w is micro-organism span time, and ξ_d is pathogen lifetime in dumpsite. Here $\alpha(t)$ is the shedding rate, $\alpha(t)_2$ is person–dumpsite contact rate. λ is dumpsite–reservoir contact rate, and μ represents the rate of birth. Additionally, r_1 is help in the DWS, r_2 is help in the DwS and r_3 is help in the MSW. Moreover, the initial conditions are $\dot{\mathcal{S}}(0) = \dot{\mathcal{S}}_0, \dot{\mathfrak{I}}(0) = \dot{\mathfrak{I}}_0, \dot{\mathcal{W}}(0) = \dot{\mathcal{W}}_0, \dot{\mathfrak{D}}(0) = \dot{\mathfrak{D}}_0$ and $\dot{\mathfrak{R}}(0) = \dot{\mathfrak{R}}_0$ depend on min $(\dot{\mathcal{S}}_0, \dot{\mathfrak{I}}_0, \dot{\mathcal{W}}_0, \dot{\mathfrak{D}}_0, \dot{\mathfrak{R}}_0) \geq 0$.

For the numerical study of (17), we presume the numerical data from [47], where $\gamma = 0.0445$, $\mu = 0.05363$, $b_l = 0.43$, $r_2 = 0.225$, $r_3 = 0.1$, $\lambda = 0.033$, $\alpha(t) = 3.89$, $\xi_w = 2.63$, $b_w = 9.49$, $b_d = 6.54$, $\xi_d = 2.58$, $r_1 = 0.335$, $\alpha(t)_2 = 19.93$, $\mathfrak{N} = 2{,}200{,}000$ Figure 1.

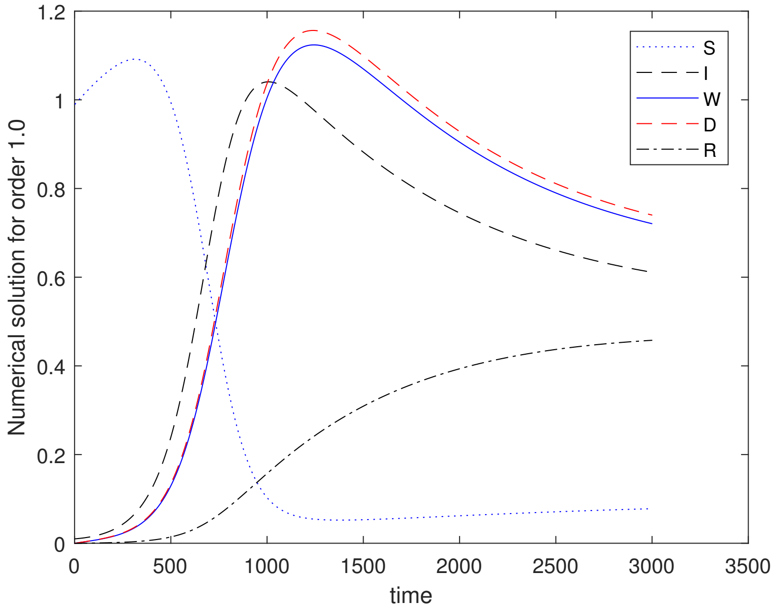

Figure 1. The computational results for the order 1 are compared with the solution for fractional order 0.99.

We can observe a rise in the classes $\dot{\mathfrak{I}}, \dot{\mathcal{W}}, \dot{\mathfrak{D}}, \dot{\mathfrak{R}}$, which can be seen in Figures 2–6, varying according to their parametric values' deviance. For the fractional orders 1.0, 0.99, 0.98, simulations are given in different graphs. These seven graphs present a comparative analysis of the simulations for the fractional constant orders with the variable orders. The variable orders have an advantage over the constant orders in that fluctuations in the dynamics can be observed, which are important for real dynamical situations. We have presumed three different fractional orders $1 - 0.006(\cos(t))^2$, $1 - 0.006(\cos(t))$ and $1 - 0.006(\sin(t))$. We have given the seventh graph i.e., Figure 7, for extending the time from 3000 units to the 3500 as per a reviewer's suggestion and can observe the fluctuations in all the classes as in the reset other plots for the variable orders.

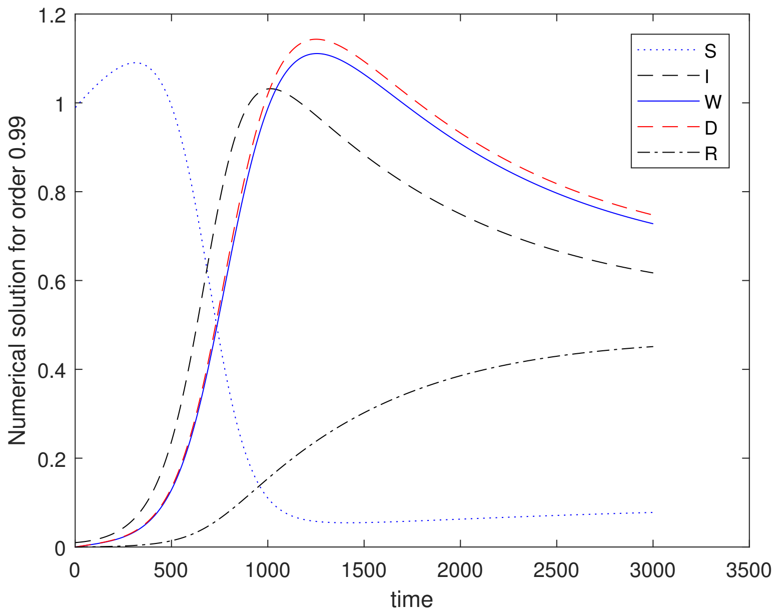

Figure 2. The computational results for the order 1 are compared with the solution for fractional order 0.99.

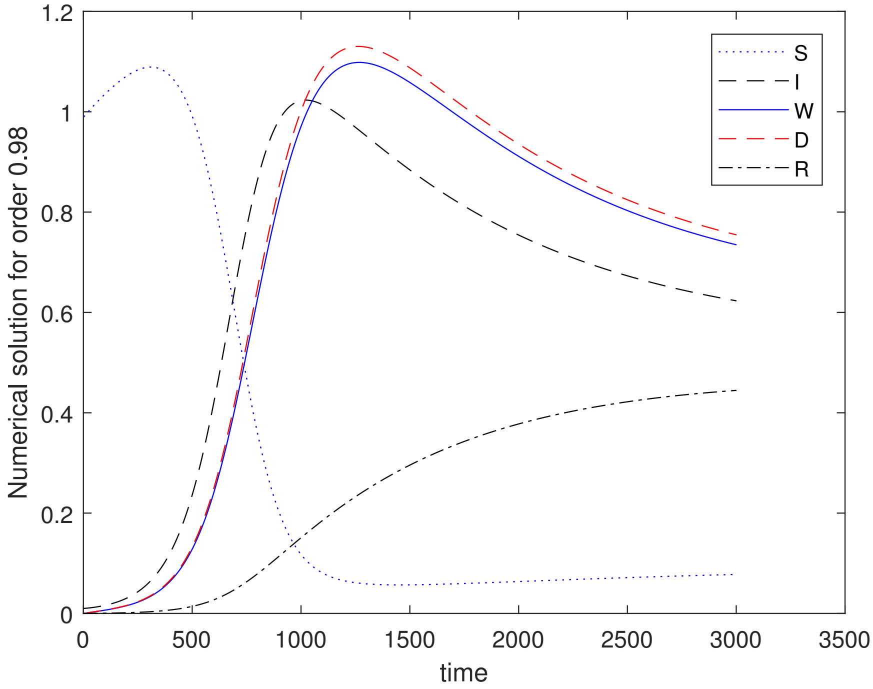

Figure 3. Numerical solution of the mathematical model for fractional order 0.98.

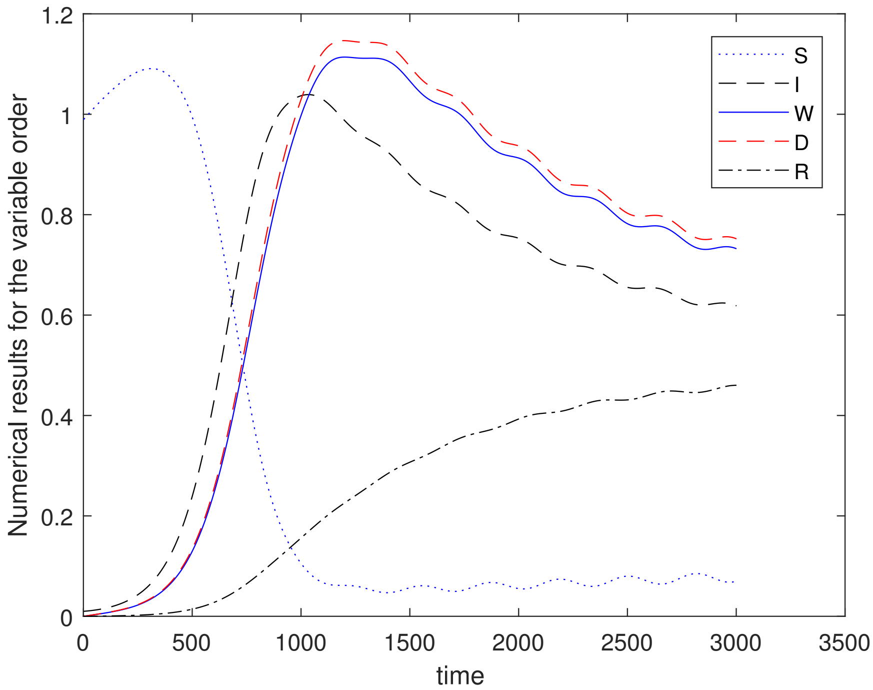

Figure 4. Numerical solution of the mathematical model for the the variable order $1 - 0.006(cos(t))^2$.

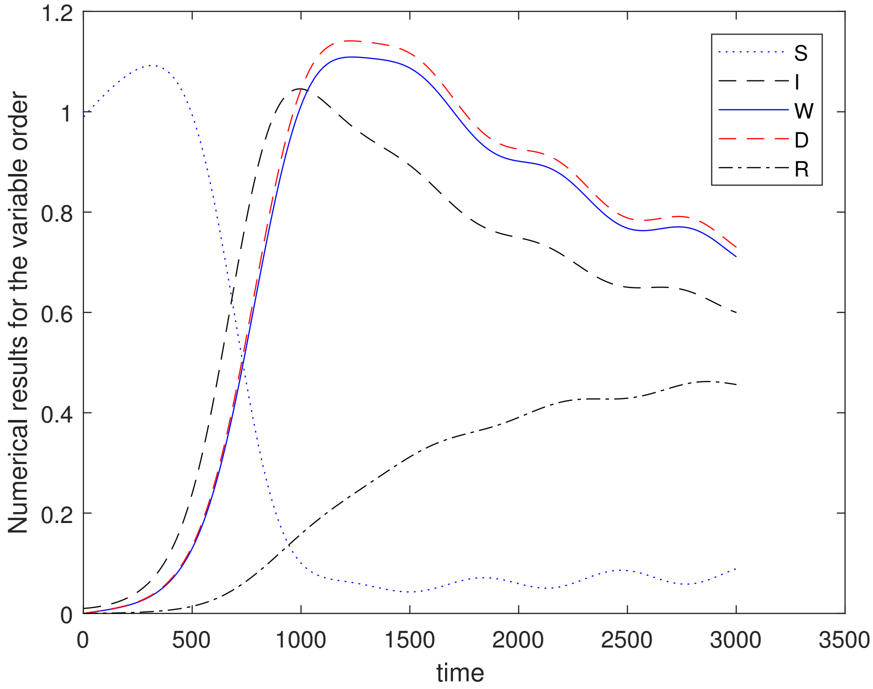

Figure 5. Numerical solution of the mathematical model (17) for the variable order $1 - 0.006 * (cos(t))$.

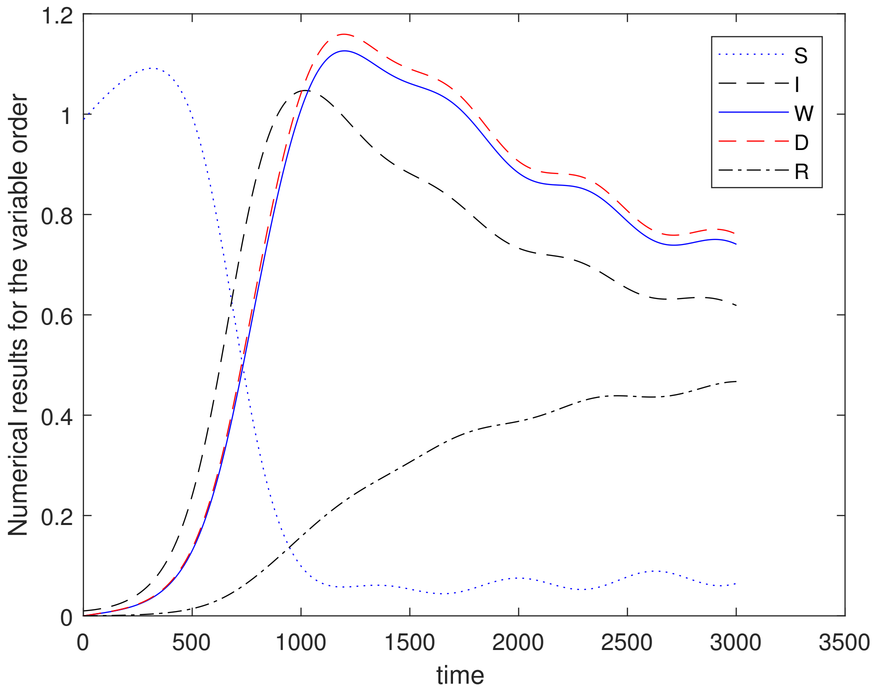

Figure 6. Numerical solution of the mathematical model (17) for the variable order $1 - 0.006 * (sin(t))$.

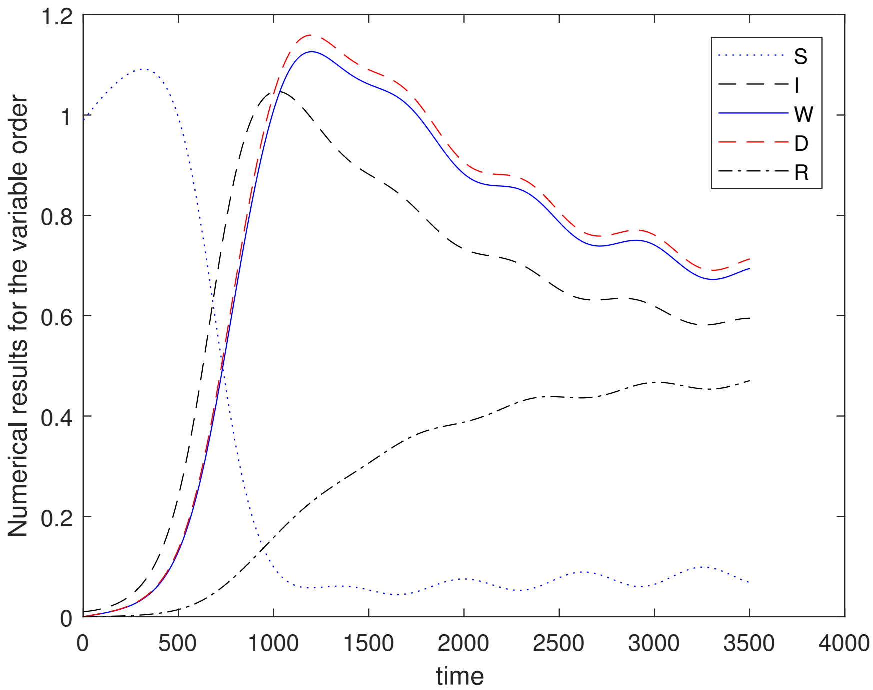

Figure 7. Numerical solution of the mathematical model (17) for the variable order $1 - 0.006 * (sin(t))$.

5. Conclusions

In this article, we have combined three basic concepts in the literature including the fractional calculus, p-Laplacian operator, and the variable order calculus and have constructed a p-Laplacian variable order hybrid system of fractional differential Equation (1). The problem (1) is a general and applied problem. There is no such construction in the literature. We have used the ABC-variable order operator and the variable order Caputo's fractional operators. The first question about this sort of problem is whether this will have a solution or not. To respond to this kind of question, we have presented the necessary and sufficient conditions for the existence and uniqueness of solutions. Hyers–Ulam stability is also presented and a numerical scheme for the simulations was produced and was applied to a waterborne disease model of the variable order. The results for the constant orders and the variable orders were compared graphically.

The variable orders have the advantage over the constant orders that the fluctuation in the dynamics can be observed which are important for real dynamical situation. We have presumed three different fractional orders $1 - 0.006(\cos(t))^2$, $1 - 0.006(\cos(t))$ and $1 - 0.006(\sin(t))$.

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