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Generalized Vector Quasi-Equilibrium Problems

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Abstract: The aim of this paper is to present new existence results for solutions to a generalized quasi-equilibrium problem with set-valued mappings and moving cones. The key to this approach is a new Browder-type fixed point theorem, which permits working in a new direction with the milder condition of transfer open-valued mapping and considering weaker assumptions on the coving cone. These results are applied to some generalized vector quasi-equilibrium problems with trifunctions and to a vector quasi-equilibrium problem with fuzzy mappings in a fuzzy environment.

Keywords: generalized vector quasi-equilibrium problem; transfer open-valued; upper $C(x)$-convex mapping; Browder-type fixed point theorem; continuous selection theorem; fuzzy mapping

MSC: 49J10; 49J53

1. Introduction

Let X and Y be two topological vector spaces, $K \subseteq X$ a nonempty subset, $S : K \to 2^K$ a set-valued mapping with nonempty values, and $C : K \to 2^Y$ a set-valued mapping such that $C(x)$ is a solid convex cone for all $x \in K$. Taking a vector-valued mapping, $h : K \times K \to Y$, the generalized vector quasi-equilibrium problem, studied in [1], consists of finding

$$\bar{x} \in S(\bar{x}) \text{ such that } h(\bar{x}, u) \in Y \setminus (-\operatorname{int} C(\bar{x})), \text{ for all } u \in S(\bar{x}),$$

and it can be extended to the set-valued mapping $H : K \times K \to 2^Y$ into the following ways:

$$(VQEP) \text{ find } \bar{x} \in S(\bar{x}) \text{ such that } H(\bar{x}, u) \subseteq Y \setminus (-\operatorname{int} C(\bar{x})), \text{ for all } u \in S(\bar{x}),$$

or

$$(VQEP1) \text{ find } \bar{x} \in S(\bar{x}) \text{ such that } H(\bar{x}, u) \nsubseteq -\operatorname{int} C(\bar{x}), \text{ for all } u \in S(\bar{x}).$$

Obviously, each solution to $(VQEP)$ is also a solution to $(VQEP1)$.

The domain of vector quasi-equilibrium problems offers a general, unified, and natural framework for the study of a large variety of problems, such as vector quasi-optimization problems, vector quasi-variational problems, and vector quasi-complementarity problems. Since the decision-maker may vary with the time, in recent years, the authors have turned their attention to various vector quasi-equilibrium problems with variable ordering structure. The ordering structure is represented by set-valued mappings whose images are cones. Existence results for the solutions to these problems were obtained by means of KKM-Fan theorem [2–8], Kakutani–Fan–Glicksberg theorem [1,6,9,10], Browder-type fixed-point theorem [4], or coincidence theorems [11]. Within these results, $C(x)$ is assumed to be a closed solid convex cone, and usually, the upper semicontinuity or the closedness of $C(\cdot)$ is demanded.

In this paper, we focus on the problem $(VQEP)$, and using a Browder-type fixed-point theorem, we provide sufficient conditions for the existence of solutions to this problem. This approach involves the set of fixed points for a given set-valued mapping, and it allows

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dealing with a set-valued cone mapping C, whose images $C(x)$ are pointed, solid, and convex cones. In Section 2, we recall some definitions and auxiliary results that we will need in this paper. Section 3 is devoted to the study of $(VQEP)$, and the main existence result is stated. The key to its proof is a recent Browder-type fixed-point theorem that involves the milder concept of transfer open value and some convexity demands. This new fixed-point theorem allows us to work in a new direction and to obtain new results for generalized vector equilibrium problems. Within the results, it is only assumed that $C(x)$ is a solid convex cone, for all $x \in K$. Then, in Section 4, we obtain some results for generalized vector quasi-equilibrium problems with trifunctions. The considered set-valued mapping T, from these problems, is assumed to take nonempty convex values, while T^{-1} is transfer open-valued on K. In Section 5, the results are applied to vector quasi-equilibrium problems with fuzzy mappings, in fuzzy environments. Our statements improve other results, and they are compared with the existing ones from the literature.

2. Preliminaries

Within this section, we recall some definitions and well-known results. From now on, X and Y are assumed to be two topological vector spaces, until otherwise is stated. Recall that a subset C of Y is called a convex cone if $tC \subseteq C$, for every $t \geq 0$, and $C + C \subseteq C$. Let Y^* be the topological dual space of Y, and let

$$C^* = \{y^* \in Y^* \mid y^*(c) \geq 0, \text{ for all } c \in C\}$$

be the positive dual cone of C. The quasi-interior of C^* is the set

$$C^\sharp = \{y^* \in C^* \mid y^*(c) > 0, \text{ for all } c \in C \setminus \{0_Y\}\}.$$

$C^\sharp$ is nonempty if and only if C has a base, i.e., there exists B such that $C = \text{cone}\,(B)$ and $0_Y \notin \overline{B}$ (see, e.g. [12]. For a nonempty set $M \subseteq K$, $\text{int}\, M$ and $\text{int}_K\, M$ stand for the interior of M and the interior of M in the relative topology of K, while $\text{co}\, M$ and $\overline{M}$ denote the convex hull and the closure of M, respectively. Given a set-valued mapping $F : K \to 2^K$, by fix F, we understand the set of all fixed points of F, i.e., $\text{fix}\, F = \{x \in K \mid x \in F(x)\}$. Note that fix F is closed whenever F is closed (see, e.g., [13]), namely $\text{Gr}\,(F) = \{(x, y) \mid x \in K, y \in F(x)\}$ is closed in $K \times K$.

Definition 1 ([3])**.** *Let K be a nonempty convex subset of X and $F : K \to 2^Y$ a set-valued mapping with nonempty values. F is said to be upper C-convex if, for any x_1, $x_2 \in K$ and any $t \in [0, 1]$, it holds that*

$$tF(x_1) + (1 - t)F(x_2) \subseteq F(tx_1 + (1 - t)x_2) + C.$$

The concept of upper C-convex mapping can be found in the literature also under the terminology of C-convex mapping (see, for instance, [14], [15] or [16]). Having a moving ordering cone, this concept can be generalized as follows.

Definition 2. *Let K be a nonempty convex subset of X, $C : K \to 2^Y$, a set-valued mapping such that $C(x)$ is a pointed solid convex cone for all $x \in K$, and $F : K \times K \to 2^Y$ a set-valued mapping with nonempty values. F is said to be upper $C(x)$-convex mapping in its second variable if, for any x, y_1, $y_2 \in K$ and any $t \in [0, 1]$, it holds that*

$$tF(x, y_1) + (1 - t)F(x, y_2) \subseteq F(x, ty_1 + (1 - t)y_2) + C(x).$$

When $C(x) = C$, then the concept of upper $C(x)$-convex mapping, reduces to the one of upper C-convex mapping of $F(x, \cdot)$, for all $x \in K$. For other concepts of generalized convexity with moving cones, we refer the reader to [4,5].

Definition 3 ([17]). *Let C be a proper convex cone and $F : X \to 2^Y$ a set-valued mapping with nonempty values. Then, F is said to be C-lower semicontinuous at $x_0 \in X$ if, for any $y \in F(x_0)$ and any open set V of y, there exists U an open neighbourhood of x_0 such that*

$$F(x) \cap (V + C) \neq \emptyset, \text{ for all } x \in U.$$

F is said to be C-lower semicontinuous on X if it is C-lower semicontinuous at each $x_0 \in X$.

In the case of single-valued mappings, we recover the C-lower semicontinuity notion, while, by replacing C with $-C$, the C-upper semicontinuity notion is recovered. The following lemma about C-lower semicontinuous mappings is stated in Lemma 2 in [16].

Lemma 1. *Let K be a nonempty subset of X and $F : K \to 2^Y$ a C-lower semicontinuous set-valued mapping on K. Then, the set*

$$\{x \in K \mid F(x) \subseteq Y \setminus \text{int } C\}$$

is closed in K.

Definition 4 ([18]). *Let K be a nonempty subset of X. A set-valued mapping $F : K \to 2^Y$ has open lower sections, if, for any $y \in Y$, the set*

$$\{x \in K \mid y \in F(x)\}$$

is open in K.

The next concept, namely the notion of transfer open value, is much weaker than the notion of the open lower section.

Definition 5 ([19]). *Let X and Y be two topological spaces. A set-valued mapping $F : X \to 2^Y$ is said to have transfer open value on X, if for every $x \in X$ and $y \in F(x)$, it implies that there exists $x' \in X$ such that $y \in \text{int } F(x')$.*

Remark 1. *Note that if a mapping $F : X \to 2^Y$ has open lower sections, then F^{-1} is transfer open-valued on Y.*

The following generalisation of the Browder fixed-point theorem was stated in Theorem 2.8 in [20].

Theorem 1. *Let K be a nonempty, convex, and compact subset of a real Hausdorff topological vector space X. Let $F : K \to 2^K$ be a set-valued mapping such that*

(i) For each $x \in K$, $F(x)$ is a nonempty convex subset of K;
(ii) F^{-1} is transfer open-valued on K.

Then, there exists $x^* \in K$ such that $x^* \in F(x^*)$.

Definition 6. *A multimap $F : X \to 2^Y$ is called compact if the closure of its range $F(X)$ is relatively compact in Y, i.e., $\overline{F(X)}$ is compact in Y.*

Lemma 2 ([21]). *Let X and Y be topological spaces, and let $F : X \to 2^Y$. Then, the following statements are equivalent:*

(i) $X = \bigcup\{\text{int } F^{-1}(y) \mid y \in Y\}$;
(ii) $F^{-1} : Y \to 2^X$ is transfer open valued on Y, and for all $x \in X$, $F(x)$ is nonempty.

To the end of this section, we recall a continuous selection theorem due to Horvath [22].

Theorem 2. *Let X be a paracompact Hausdorff space, Y a convex space, $G, T : X \to 2^Y$ two set-valued mappings such that* co $G(x) \subseteq T(x)$*, for any $x \in X$, and $X = \bigcup\{\text{int}\, G^{-1}(y) | \, y \in Y\}$. Then, T has a continuous selection $f : X \to Y$, that is, $f(x) \in T(x)$ for all $x \in X$.*

3. Main Section

Within this section, sufficient conditions for the existence of solutions to the problem $(VQEP)$ are provided, by using a technique relying on a Bowder-type fixed-point theorem.

Theorem 3. *Let K be a nonempty, compact, and convex subset of X, $C : K \to 2^Y$ a set-valued mapping such that $C(x)$ is a pointed solid convex cone for all $x \in K$, $H : K \times K \to 2^Y$ a set-valued mapping with nonempty values, and $S : K \to 2^K$ a mapping with nonempty convex values such that* fix S *is closed. Assume that the following conditions are satisfied:*

(i) $H(x, x) \subseteq Y \setminus (-\text{int}\, C(x))$ *for all* $x \in \text{fix}\, S$*;*
(ii) $\{u \in K | \, H(x, u) \not\subseteq Y \setminus (-\text{int}\, C(x))\}$ *is convex for all* $x \in \text{fix}\, S$*;*
(iii) S^{-1} *is transfer open valued on K;*
(iv) *The mapping $u \in K \mapsto \{x \in K | \, H(x, u) \not\subseteq Y \setminus (-\text{int}\, C(x))$, and $u \in S(x)\}$ is transfer open-valued on K.*

Then, the problem $(VQEP)$ is solvable.

Proof. The set-valued mapping S fulfils all the assumptions of Theorem 1, and therefore, fix $S \neq \emptyset$. With the aid of fix S, we construct a set-valued mapping $G : K \to 2^K$ as follows:

$$G(x) = \begin{cases} F(x) \cap S(x), \text{ if } x \in \text{fix}\, S, \\ S(x), \text{ if } x \notin \text{fix}\, S, \end{cases}$$

where $F : K \to 2^K$ is defined as

$$F(x) = \{u \in K \mid \; H(x, u) \not\subseteq Y \setminus (-\text{int}\, C(x))\}.$$

For every $x \in K$, $G(x)$ is a convex set due to F and S set-valued mappings with convex values. By contradiction, suppose that G has nonempty values. In order to prove that G^{-1} is transfer open valued on K, let $u \in K$. Thus,

$$\begin{aligned} G^{-1}(u) &= \{x \in K \mid \; u \in G(x)\} \\ &= \{x \in \text{fix}\, S \mid \; u \in (F \cap S)(x)\} \cup \{x \in K \setminus \text{fix}\, S \mid \; u \in S(x)\} \\ &= \Big(\text{fix}\, S \cap (F \cap S)^{-1}(u)\Big) \cup \Big((K \setminus \text{fix}\, S) \cap S^{-1}(u)\Big), \end{aligned}$$

and, by the distribution rule of union, it comes out that $G^{-1}(u)$ equals

$$\begin{aligned} &= \Big[\Big(\text{fix}\, S \cap (F \cap S)^{-1}(u)\Big) \cup \Big(K \setminus \text{fix}\, S\Big)\Big] \cap \Big[\Big(\text{fix}\, S \cap (F \cap S)^{-1}(u)\Big) \cup S^{-1}(u)\Big] \\ &= \Big[K \cap \Big((F \cap S)^{-1}(u) \cup (K \setminus \text{fix}\, S)\Big)\Big] \cap \Big[\Big(\text{fix}\, S \cup S^{-1}(u)\Big) \cap \Big((F \cap S)^{-1}(u) \\ &\quad \cup S^{-1}(u)\Big)\Big]. \end{aligned}$$

Because $(F \cap S)(x) \subseteq S(x)$ for all $x \in K$, $(F \cap S)^{-1}(u) \subseteq S^{-1}(u)$ for any $u \in K$. The above inclusion and the distribution rule of intersection provide that

$$\begin{aligned} G^{-1}(u) &= \Big[K \cap \Big((F \cap S)^{-1}(u) \cup (K \setminus \text{fix}\, S)\Big)\Big] \cap \Big[\Big(\text{fix}\, S \cup S^{-1}(u)\Big) \cap S^{-1}(u)\Big] \\ &= \Big[(F \cap S)^{-1}(u) \cup (K \setminus \text{fix}\, S)\Big] \cap S^{-1}(u) \\ &= \Big[(F \cap S)^{-1}(u)\Big] \cup \Big[(K \setminus \text{fix}\, S) \cap S^{-1}(u)\Big]. \end{aligned}$$

Now, let $x \in G^{-1}(u)$. The following cases are distinguished.

(i) If $x \in (F \cap S)^{-1}(u)$, then due to assumption (iv), there exists $u' \in K$ such that

$$x \in \text{int}_K (F \cap S)^{-1}(u') \subseteq \text{int}_K G^{-1}(u').$$

(ii) If $x \in (K \setminus \text{fix } S) \cap A^{-1}(u)$, we have that $x \in K \setminus \text{fix } S$, which is an open set in the relative topology on K, and that $x \in S^{-1}(u)$. Assumption (ii) assures the existence of an element $u'' \in K$ such that $x \in \text{int}_K S^{-1}(u'')$, and therefore,

$$x \in \text{int}_K \left((K \setminus \text{fix} S) \cap S^{-1}(u'') \right) \subseteq \text{int}_K G^{-1}(u'').$$

So, G^{-1} is transfer open-valued on K.

Since G verifies all the assumptions of Theorem 1, then there exists $x^* \in K$ such that $x^* \in G(x^*)$. The following cases hold.

(i) If $x^* \in \text{fix } S$, then $x^* \in F(x^*) \cap S(x^*)$, wherefrom $x^* \in F(x^*)$, i.e., $H(x^*, x^*) \not\subseteq Y \setminus (-\text{int } C(x^*))$, which contradicts condition (i).

(ii) If $x^* \in K \setminus \text{fix } S$, then $x^* \in S(x^*)$, i.e., $x^* \in \text{fix } S$, which is impossible.

Thus, the conjecture made is false, and therefore, there exists at least one $\bar{x} \in K$ such that $G(\bar{x}) = \emptyset$. If

(i) $\bar{x} \in \text{fix } S$, then $F(\bar{x}) \cap S(\bar{x}) = \emptyset$. So, there exists $\bar{x} \in \text{fix } S$ such that

$$H(\bar{x}, u) \subseteq Y \setminus (-\text{int } C(\bar{x})), \text{ for all } u \in S(\bar{x}).$$

(ii) $\bar{x} \in K \setminus \text{fix } S$, then $S(\bar{x}) = \emptyset$ it is a contradiction to the fact that S takes nonempty values.

Therefore, there exists $\bar{x} \in S(\bar{x})$ such that

$$H(\bar{x}, u) \subseteq Y \setminus (-\text{int } C(\bar{x})), \text{ for all } u \in S(\bar{x}).$$

□

By taking $C(x) = C$, for all $x \in K$, in Theorem 3, then Theorem 4 represents an improvement of Theorem 3.1 in [23] in the sense that $H(x, x) \subseteq Y \setminus (-\text{int } C)$ is required only for all $x \in \text{fix } S$ and not for all the elements from the set K. We notice that Theorem 3 also provides sufficient conditions for the existence of solutions to the problem $(VQEP1)$. Although the weaker transfer open concept is used, Theorem 3 is different from Corollary 3.5 in [3], Theorem 3.1 in [1], Theorem 3.1 in [5], and Theorems 3.1 and 3.2 in [7], approaches relied on by other theorems, where pseudomonotonicity assumptions or other generalized convexity concepts are assumed.

Theorem 4. *Let K be a nonempty, compact, and convex subset of X, $H : K \times K \to 2^Y$ a set-valued mapping with nonempty values, and $S : K \to 2^K$ a mapping with nonempty convex values such that* fix S *is closed. Assume that the following conditions are satisfied:*

(i) $H(x, x) \subseteq Y \setminus (-\text{int } C)$ *for all* $x \in \text{fix } S$*;*

(ii) $\{u \in K | H(x, u) \not\subseteq Y \setminus (-\text{int } C)\}$ *is convex for all* $x \in \text{fix } S$*;*

(iii) S^{-1} *is transfer open valued on K;*

(iv) The mapping $u \in K \mapsto \{x \in K | H(x, u) \not\subseteq Y \setminus (-\text{int } C)$ *and* $u \in S(x)\}$ *is transfer open-valued on K.*

Hence, there exists $\bar{x} \in S(\bar{x})$ such that

$$H(\bar{x}, u) \subseteq Y \setminus (-\text{int } C), \textit{ for all } u \in S(\bar{x}).$$

Proof. Take in Theorem 3, $C(x) = C$ for all $x \in K$. □

The next statement provides some conditions under which the assumptions of Theorem 3 are satisfied.

Corollary 1. *Let $K \subseteq X$ be a nonempty, compact, and convex subset of X, $C : K \to 2^Y$ a set-valued mapping such that $C(x)$ is a pointed solid convex cone for all $x \in K$, $H : K \times K \to 2^Y$ a set-valued mapping with nonempty values, and $S : K \to 2^K$ a nonempty convex valued mapping with open lower sections such that* fix S *is closed. Assume that the following conditions are satisfied:*

(i) $H(x,x) \subseteq Y \setminus (-\text{int}\, C(x))$ *for all* $x \in \text{fix}\, S$;
(ii) *For every* $u \in K$, $\{x \in K | \, H(x,u) \nsubseteq Y \setminus (-\text{int}\, C(x))\}$ *is an open set in* K;
(iii) *For every* $x \in K$, H *is an upper* $C(x)$*, mapping its second variable.*

Therefore, problem $(VQEP)$ *is solvable.*

Proof. Since S has open lower sections, then S^{-1} is transfer open-valued on K and the set

$$\{x \in K \mid \ H(x,u) \nsubseteq Y \setminus (-\text{int}\, C(x)) \text{ and } u \in S(x)\} =$$

$$= \{x \in K \mid \ H(x,u) \nsubseteq Y \setminus (-\text{int}\, C(x))\} \cap S^{-1}(u)$$

is open, for all $u \in K$, and thus conditions (ii) and (iv) of Theorem 3 are verified. It remains for us to check the convexity of $\{u \in K | \, H(x,u) \nsubseteq Y \setminus (-\text{int}\, C(x))\}$, for all $x \in \text{fix}\, S$. So, let $u_1, u_2 \in K$ and $t \in [0,1]$ such that

$$H(x,u_1) \nsubseteq Y \setminus (-\text{int}\, C(x)) \text{ and } H(x,u_2) \nsubseteq Y \setminus (-\text{int}\, C(x)).$$

Because of this, there exists $z_1 \in H(x,u_1) \cap (-\text{int}\, C(x))$ and $z_2 \in H(x,u_2) \cap (-\text{int}\, C(x))$ such that $tz_1 + (1-t)z_2 \in -\text{int}\, C(x)$ and

$$tz_1 + (1-t)z_2 \in tH(x,u_1) + (1-t)H(x,u_2) \subseteq H(x, tu_1 + (1-t)u_2) + C(x).$$

Therefore, we conclude that

$$H(x, tu_1 + (1-t)u_2) \cap (-\text{int}\, C(x)) \neq \emptyset,$$

and so $\{u \in K | \, H(x,u) \nsubseteq Y \setminus (-\text{int}\, C(x))\}$ is a convex set. □

Corollary 2. *Let $K \subseteq X$ be a nonempty, compact, and convex subset, $H : K \times K \to 2^Y$ a set-valued mapping with nonempty values, and $S : K \to 2^K$ a nonempty convex valued mapping with open lower sections such that* fix S *is closed. Assume that the following conditions are satisfied:*

(i) $H(x,x) \subseteq Y \setminus (-\text{int}\, C)$ *for all* $x \in \text{fix}\, S$;
(ii) *For every* $u \in K$, *the set-valued mapping* $x \mapsto H(x,u) : K \to 2^Y$ *is* $-C$ *lower semicontinuous on* K;
(iii) *For every* $x \in K$, *the set-valued mapping* $u \mapsto H(x,u) : K \to 2^Y$ *is an upper* C *mapping.*

Hence, there exists $\bar{x} \in S(\bar{x})$ *such that*

$$H(\bar{x},u) \subseteq Y \setminus (-\text{int}\, C), \textit{ for all } u \in S(\bar{x}).$$

Proof. In order to apply Corollary 1, with $C(x) = C$ for all $x \in K$, it remains for us to check assumption (ii). For every $u \in K$, by Lemma 1, the set $\{x \in K | \, H(x,u) \subseteq Y \setminus (-\text{int}\, C)\}$ is closed, wherefrom

$$\{x \in K | \, H(x,u) \nsubseteq Y \setminus (-\text{int}\, C)\}$$

is an open set in K. □

By taking $X = Z$ in Theorem 2 in [11], for the above problem, an existence result is obtained, but for the particular case in which $S(x) = K$, for all $x \in K$. Even if we consider the milder concept of $-C$-lower semicontinuouity in the first variable instead of the lower semicontinuity assumption, Corollary 2 is different from Theorem 12 in [11], where other concepts of generalized convexity are used and other types of assumptions are made.

4. Applications to Generalized Vector Quasi-Equilibrium Problems with Trifunctions

In the sequel, the results are applied to $(QVEP2)$, a generalized vector quasi-equilibrium problem with set-valued mappings, which consists of finding

$$\bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x}) \text{ such that } G(\bar{x}, \bar{y}, u) \subseteq Y \setminus (-\text{int } C), \forall u \in S(\bar{x}),$$

where D is a nonempty subset of a topological space, and $S : K \to 2^K$, $T : K \to 2^D$ and $G : K \times D \times K \to 2^Y$ are set-valued mappings.

Theorem 5. *Let K be a nonempty, convex, and compact subset of a complete Hausdorff locally convex space X, D a convex subset of a topological vector space, $T : K \to 2^D$ a set-valued mapping with nonempty convex values, $S : K \to 2^K$ a compact set-valued mapping with nonempty convex values, open lower sections such that* fix S *is closed. If $G(x, y, x) \subseteq Y \setminus (-\text{int } C)$, for all $(x, y) \in \text{fix } S \times T(K)$ and the following conditions are satisfied:*

(i) $K = \bigcup \{\text{int } T^{-1}(d) \,|\, d \in D\}$;

(ii) *For all $(x, y) \in \text{fix } S \times T(K)$, the set $\{u \in K \mid G(x, y, u) \nsubseteq Y \setminus (-\text{int } C)\}$ is convex;*

(iii) *G is $-C$-lsc in its first and second variable;*

then, the problem $(QVEP2)$ is solvable.

Proof. Firstly, a continuous selection for T is provided, and with its help a new set-valued mapping is introduced. As T has convex values, then co $T(x) = T(x)$. By Lemma 2 and Theorem 2, there exists $f : K \to D$, a continuous function such that $f(x) \in T(x)$, for all $x \in K$.

Further, S compact, gives $\overline{S(K)}$ compact set in K. Therefore, according to Theorem 4.8.9 in [24], the set

$$K' = \overline{\text{co}}\, \overline{S(K)}$$

is convex and compact.

Consider now the auxiliary set-valued mapping $H : K' \times K' \to 2^{K'}$, defined by

$$H(x, u) = G(x, f(x), u), \ \forall (x, u) \in K' \times K'.$$

As K' is a convex subset of K, by assumption (ii), the set

$$\{u \in K' |\, G(x, f(x), u) \nsubseteq Y \setminus (-\text{int } C)\} = \{u \in K' |\, H(x, u) \nsubseteq Y \setminus (-\text{int } C)\}$$

is convex for all $x \in \text{fix } S$. In order to apply Theorem 4, it remains for us to prove that H is $-C$-lsc on K' in its first variable. So, let $x_0 \in K'$, and let $z_0 \in H(x_0, u) = G(x_0, f(x_0), u)$. Further, let $V \in \mathcal{N}(z_0)$. By the $-C$-lsc of G in its first variable, there exists $U_1 \in \mathcal{N}(x_0)$ such that

$$G(x, f(x_0), u) \cap (V - C) \neq \emptyset, \ \forall x \in U_1.$$

Consequently, there exists $z \in G(x, f(x_0), u)$ such that $z \in V - C$. As $V - C \in \mathcal{N}(z)$, by the $-C$-lsc of G in its second variable, there exists $W \in \mathcal{N}(f(x_0))$ such that

$$G(x, y, u) \cap (V - C - C) \neq \emptyset, \ \forall y \in W.$$

By the continuity of f and $W \in \mathcal{N}(f(x_0))$, there exists $U_2 \in \mathcal{N}(x_0)$ such that $f(x) \in W$, for all $x \in U_2$. Taking now $U = U_1 \cap U_2 \in \mathcal{N}(x_0)$, it takes place as follows:

$$G(x, f(x), u) \cap (V - C) \neq \emptyset, \forall x \in U,$$

namely H is $-C$-lsc in its first variable.

As $G(x,y,x) \subseteq Y \setminus (-\text{int } C)$, for all $(x,y) \in \text{fix } S \times T(K)$, we obtain $H(x,x) \subseteq Y \setminus (-\text{int } C)$ for all $x \in \text{fix } S$. Thus, all the assumptions of Theorem 4 are verified, and so there exist $\bar{x} \in S(\bar{x})$ and $\bar{y} = f(\bar{x}) \in T(\bar{x})$ such that

$$G(\bar{x},\bar{y},u) \subseteq Y \setminus (-\text{int } C), \forall u \in S(\bar{x}).$$

□

The convexity hypothesis (ii) of Theorem 5 is replaced in the following corollary by the upper C mapping demand on G in its third variable.

Corollary 3. *Let K be a nonempty, convex, and compact subset of a complete Hausdorff locally convex space X, D a convex subset of a topological vector space, $T : K \to 2^D$ a set-valued mapping with nonempty convex values, and $S : K \to 2^K$ a compact set-valued mapping with nonempty convex values, open lower sections such that* fix S *is closed. If $G(x,y,x) \subseteq Y \setminus (-\text{int } C)$, for all $(x,y) \in \text{fix } S \times T(K)$ and the following conditions are satisfied:*

(i) $K = \bigcup\{\text{int } T^{-1}(d) | \, d \in D\}$;
(ii) *G is an upper C-mapping in its third variable;*
(iii) *G is $-C$-lsc in its first and second variable;*

then, the problem $(QVEP2)$ is solvable.

Proof. Condition (ii) implies assumption (ii) of Theorem 5, and the conclusion follows now by this. □

Remark 2. *In Theorem 7 in [25], the authors stated an existence result for $(QVEP2)$, by considering some acyclic assumptions and that G is a continuous set-valued mapping on its domain. Corollary 3 has the milder $-C$-lower semicontinuity assumption on G, and the additional upper C-mapping assumption. Moreover, instead of $G(x,y,x) \subseteq C$ for all $(x,y) \in K \times D$, we have a much weaker demand, namely $G(x,y,x) \subseteq Y \setminus (-\text{int } C)$, for all $(x,y) \in \text{fix } S \times T(K)$.*

The next lemma allows obtaining a new statement for the existence of solutions to the problem $(QVEP2)$. Within this result, the compactness assumption on S is dropped, and in addition, the upper semicontinuity of S is demanded.

Lemma 3 ([26])**.** *Let X be a real topological vector space, $S : X \to 2^X$ be upper semicontinuous on X, and K a compact subset of X. Then, $S(K)$ is compact.*

Theorem 6. *Let K be a nonempty, convex, and compact subset of a complete Hausdorff locally convex space X, D a convex subset of a topological vector space, $T : K \to 2^D$ a set-valued mapping with nonempty convex values, and $S : K \times K \to 2^K$ an upper semicontinuous set-valued mapping on K with nonempty convex values, open lower sections such that* fix S *is closed. If $G(x,y,x) \subseteq Y \setminus (-\text{int } C)$, for all $(x,y) \in \text{fix } S \times T(K)$ and the following conditions are satisfied:*

(i) $K = \bigcup\{\text{int } T^{-1}(d) | \, d \in D\}$;
(ii) *For all $(x,y) \in \text{fix } S \times T(K)$, the set $\{u \in K \mid G(x,y,u) \not\subseteq Y \setminus (-\text{int } C)\}$ is convex;*
(iii) *G is $-C$-lsc in its first and second variable;*

then, problem $(QVEP2)$ is solvable.

Proof. It follows by Lemma 3, and it is similar to the proof of Theorem 5. □

Taking now $S(x) = K$, for all $x \in K$, the next result holds, and it also provides sufficient conditions for the existence of solutions to the vector quasi-equilibrium problem considered in [8]. Although conditions (ii) and (iii) appear in Theorem 3.1 in [8], where a coercivity condition replaces the compactness assumption on K, some additional assumptions are demanded on G, and different assumptions are considered on T.

Theorem 7. *Let K be a nonempty, convex, and compact subset of a complete Hausdorff locally convex space X, D a convex subset of a topological vector space, and $T : K \to 2^D$ a set-valued mapping with nonempty convex values. If $G(x,y,x) \subseteq Y \setminus (-\text{int}\, C)$, for all $(x,y) \in K \times T(K)$ and the following conditions are satisfied:*

(i) $K = \bigcup\{\text{int}\, T^{-1}(d) | \, d \in D\}$;
(ii) *G is an upper C-mapping in its third variable;*
(iii) *G is $-C$-lsc in its first and second variable;*

then, there exists $\bar{x} \in K$, $\bar{y} \in T(\bar{x})$ such that

$$G(\bar{x},\bar{y},u) \subseteq Y \setminus (-\text{int}\, C), \forall u \in K.$$

Under the condition that C is a based cone, by Corollary 3, the solutions are obtained to another generalized vector quasi-equilibrium problem.

Theorem 8. *Let C be a based cone, K a nonempty convex and compact subset of a complete Hausdorff locally convex space X, D a convex subset of a topological vector space, $T : K \to 2^D$ a set-valued mapping with nonempty convex values, and $S : K \times 2^K$ a compact set-valued mapping with nonempty convex values, open lower sections such that* fix S *is closed. If $G(x,y,x) \subseteq C$, for all $(x,y) \in$* fix $S \times T(K)$ *and the following conditions are satisfied:*

(i) $K = \bigcup\{\text{int}\, T^{-1}(d) | \, d \in D\}$;
(ii) *G is an upper C-mapping in its third variable;*
(iii) *G is $-C$-lsc in its first and second variable;*

then, there exists $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ such that

$$G(\bar{x},\bar{y},u) \subseteq Y \setminus (-C \setminus \{0_Y\}), \forall u \in S(\bar{x}).$$

Proof. Since C is a based cone, then $C^\sharp \neq \emptyset$. Let $c^* \in C^\sharp$ and define

$$\overline{C} = \{y \in Y | c^*(y) > 0\} \cup \{0_Y\}.$$

It is easy to see that, $\overline{C}$ is a solid convex cone, with the property that $C \setminus \{0_Y\} \subseteq \text{int}\, \overline{C}$. Since $G(x,y,x) \subseteq C$, it implies $G(x,y,x) \subseteq \overline{C}$, and therefore, $G(x,y,x) \subseteq Y \setminus (-\text{int}\, \overline{C})$. As all the assumptions of Corollary 3 are verified, there exist $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$G(\bar{x},\bar{y},u) \subseteq Y \setminus (-\text{int}\, \overline{C}), \forall u \in S(\bar{x}).$$

wherefrom

$$G(\bar{x},\bar{y},u) \subseteq Y \setminus (-C \setminus \{0_Y\}), \forall u \in S(\bar{x}).$$

□

5. An Application to a Vector Quasi-Equilibrium Problem with Fuzzy Mappings in a Fuzzy Environment

The theory of fuzzy sets started in 1965 with the publication by L. Zadeh [27]. Fuzzy logic allows decision making with estimated values under incomplete or uncertain information. Since then, the attention of the researchers also turned to generalized games and equilibrium problems with fuzzy mappings and their particular cases (see, for instance [28–32] and the references therein).

A fuzzy set on K is a function with domain K and values in $[0,1]$. By $\mathcal{F}(K)$, we denote the collection of all fuzzy sets on K. A mapping $f : K \to \mathcal{F}(K)$ is called a fuzzy mapping, and we denote $f(x) := f_x$, for all $x \in K$, which is a fuzzy set on K. For any $u \in K$, $f_x(u) \in [0,1]$ is called the degree of membership of u in f_x, while the set

$$(f_x)_\lambda = \{u \in K | \, f_x(u) > \lambda\}$$

is called the strict λ-cut set of f_x, with $\lambda \in [0,1)$ (see, for instance [29,30]).

Definition 7 ([29]). *A fuzzy mapping $f : K \to \mathcal{F}(K)$ is called convex, if for each $x \in K$, the fuzzy set f_x is a fuzzy convex set, i.e., for any $u_1, u_2 \in K$, for all $t \in [0, 1]$,*

$$f_x(tu_1 + (1-t)u_2) \geq \min\{f_x(u_1), f_x(u_2)\}.$$

Observe that whenever f is a convex fuzzy mapping, then for any $x \in K$, the strong λ-cuts of f_x are convex sets, for all $\lambda \in [0, 1)$. Let $h : K \to \mathbb{R}$ be a real valued function. We say that h is upper semicontinuous on K, if for each $t \in \mathbb{R}$, the set $\{x \in K \mid h(x) < t\}$ is open in K. We say that h is lower semicontinuous on K if $-h$ is upper semicontinuous on K, so that $\{x \in K \mid h(x) > t\}$ is open in K, for each $t \in \mathbb{R}$.

Theorem 9. *Let K be a nonempty, convex, and compact subset of a complete Hausdorff locally convex space X, D a nonempty convex subset of a topological vector space, $s : K \to \mathcal{F}(K)$, $t : K \to \mathcal{F}(D)$ and $g : K \times D \times K \to \mathcal{F}(Y)$ be fuzzy mappings, $\alpha, \beta, \gamma : K \to [0, 1)$ such that α and γ are upper semicontinuous functions on K, and the following conditions satisfied:*

(i) For each $x \in K$, $(s_x)_{\alpha(x)}$ is nonempty, convex, $\{x \in K | x \in (s_x)_{\alpha(x)}\}$ is closed, while $\overline{\bigcup_{x\in K}(s_x)_{\alpha(x)}}$ is compact in K, and for all $u \in K$ the function $x \mapsto s_x(u)$ is lower semicontinuous as a function of x;

(ii) For each $x \in K$, $(t_x)_{\beta(x)}$ is nonempty convex, and $K = \bigcup_{d\in D} \operatorname{int}\{x \in K | d \in (t_x)_{\beta(x)}\}$;

(iii) For all $z \in Y$, the functions $x \mapsto g_{x,y,u}(z)$ and $y \mapsto g_{x,y,u}(z)$ are lower semicontinuous as a function of x, respectively, y, and the set

$$\{u \in K | (g_{x,y,u})_{\gamma(x)} \not\subseteq Y \setminus (-\operatorname{int} C)\}$$

is convex for all $(x, y) \in K \times D$;

(iv) $\{z \in Y \mid g_{x,y,x}(z) > \gamma(x)\} \subseteq Y \setminus (-\operatorname{int} C)$, for all $x \in \{x \in K | x \in (s_x)_{\alpha(x)}\}$ and $y \in D$.

Hence, there exists $\bar{x} \in (s_{\bar{x}})_{\alpha(\bar{x})}$, $\bar{y} \in (t_{\bar{x}})_{\beta(\bar{x})}$ such that

$$(g_{\bar{x},\bar{y},u})_{\gamma(\bar{x})} \subseteq Y \setminus (-\operatorname{int} C), \forall u \in (s_{\bar{x}})_{\alpha(\bar{x})}.$$

Proof. Define the following set-valued mappings

$$S : K \to 2^K,\ T : K \to 2^D \text{ and } G : K \times D \times K \to 2^Y$$

as $S(x) = (s_x)_{\alpha(x)} = \{u \in K \mid s_x(u) > \alpha(x)\}$, $T(x) = (t_x)_{\beta(x)} = \{d \in D \mid t_x(d) > \beta(x)\}$, and $G(x, y, u) = (g_{x,y,u})_{\gamma(x)} = \{z \in Y \mid g_{x,y,u}(z) > \gamma(x)\}$.

By assumption (i), $S(x)$ is nonempty and convex for all $x \in K$. For all $u \in K$, we have that the function $x \mapsto s_x(u) - \alpha(x)$ is lower semicontinuous as a function of x, and therefore, the set

$$\{x \in K \mid s_x(u) - \alpha(x) > 0\}$$

is open in K. Moreover,

$$S^{-1}(u) = \{x \in K | u \in S(x)\} = \{x \in K | s_x(u) > \alpha(x)\}$$

is open, i.e., S has open lower sections. The fixed point set of S, namely

$$\operatorname{fix} S = \{x \in K | x \in S(x)\} = \{x \in K | s_x(x) > \alpha(x)\}$$

is closed, while

$$\overline{S(K)} = \overline{\bigcup_{x\in K}(s_x)_{\alpha(x)}}$$

is compact, which means that S is compact.

Assumption (ii) provides that T has nonempty convex values and that

$$K = \bigcup_{d \in D} \text{int}\{x \in K |\, d \in T(x)\} = \bigcup_{d \in D} \text{int}\, T^{-1}(d).$$

Let us check now the $-C$-lsc of G in its first and second variable. For this, let $x_0 \in K$, $z \in G(x_0, y, u)$, and V is an open neighbourhood of z. Therefore $g_{x_0,y,u}(z) > \gamma(x_0)$ and by the lower semicontinuity of $x \mapsto g_{x,y,u}(z) - \gamma(x)$ as a function of x, for $\epsilon = g_{x_0,y,u}(z) - \gamma(x_0) > 0$, we obtain the existence of a neighbourhood U of x_0 such that

$$g_{x,y,u}(z) - \gamma(x) - g_{x_0,y,u}(z) + \gamma(x_0) > -\epsilon,\ \forall x \in U \Rightarrow g_{x,y,u}(z) > \gamma(x),\ \forall x \in U.$$

From, $z \in G(x, y, u)$ for all $x \in U$ and $z \in V - C$ it yields

$$G(x, y, u) \cap (V - C) \neq \emptyset,\ \forall x \in U.$$

Now, let $y_0 \in K, z \in G(x, y_0, u)$, and V an open neighbourhood of z. So, $g_{x,y_0,u}(z) > \gamma(x)$, and by the lower semicontinuity of $y \mapsto g_{x,y,u}(z)$ as a function of y, choosing $\epsilon = g_{x,y_0,u}(z) - \gamma(x) > 0$, we obtain the existence of a neighbourhood U of y_0 such that

$$g_{x,y,u}(z) - g_{x,y_0,u}(z) > -\epsilon,\ \forall y \in U \Rightarrow g_{x,y,u}(z) > \gamma(x),\ \forall y \in U.$$

Consequently $z \in G(x, y, u)$ for all $y \in U$, and together with $z \in V - C$, it provides

$$G(x, y, u) \cap (V - C) \neq \emptyset,\ \forall y \in U.$$

Assumption (iv) assures that $G(x, y, x) \subseteq Y \setminus (-\text{int}\, C)$, for all $(x, y) \in \text{fix}\, S \times D$, and in particular, this holds for all $(x, y) \in \text{fix}\, S \times T(K)$. As all the assumptions of Theorem 5 are verified, the proof is completed. □

Remark 3. *If, for any $d \in D$, the function $x \to t_x(d)$ is lower semicontinuous as a function of x, and $\beta : K \to [0, 1)$ is upper semicontinuous, then T has open lower sections. Taking into consideration Remark 1 and Lemma 1, assumption*

$$K = \bigcup_{d \in D} \text{int}\, \{x \in K |\, d \in (t_x)_{\beta(x)}\}$$

from condition (ii) of Theorem 9, can be replaced by the above semicontinuity assumptions.

The following example illustrates that all the assumptions of Theorem 9 are satisfied.

Example 1. *Let $K = D = [0, 2]$ be nonempty, convex, and compact sets, $Y = \mathbb{R}$, $C = [0, \infty)$, $s : K \to \mathcal{F}(K)$, $t : K \to \mathcal{F}(D)$, $g : K \times D \times K \to \mathcal{F}(Y)$ be fuzzy mappings defined by*

$$s_x(u) = \begin{cases} 1 & x \in [0,1], u \in (1,2]; \\ \frac{u+x^2+2}{4} & x \in [0,1], u \in [0,1]; \\ 1 & x \in (1,2], u \in (1,2]; \\ \frac{u+3}{4}, & x \in (1,2], u \in [0,1], \end{cases} \quad t_x(d) = \frac{1}{1+2d}, \forall x \in [0,2], d \in [0,2],$$

$$g_{x,y,u}(z) = \begin{cases} \frac{1}{2+2z+\frac{y^2}{2}} & x \in [0,2], y \in [0,2], u \in [0,2], z \in [0,10]; \\ \frac{1}{15} & x \in [0,2], y \in [0,2], u \in [0,2], z \in \mathbb{R} \setminus [0,10], \end{cases}$$

and, let $\alpha, \beta, \gamma : K \to [0, 1)$ be defined by

$$\alpha(x) = \begin{cases} \frac{x^2+1}{4} & x \in [0,1]; \\ \frac{1}{2} & x \in (1,2], \end{cases} \quad \beta(x) = \frac{1}{7+x^2}, \gamma(x) = \frac{1}{6+2x^2}, \forall x \in [0,2].$$

Taking any $x \in [0,1]$, we have

$$
\begin{aligned}
S(x) &= \{u \in [0,2] \mid s_x(u) > \alpha(x)\} = \{u \in [0,2] \mid s_x(u) > \frac{x^2+1}{4}\} \\
&= \{u \in (1,2] \mid 1 > \frac{x^2+1}{4}\} \cup \{u \in [0,1] \mid \frac{u+x^2+2}{4} > \frac{x^2+1}{4}\} \\
&= (1,2] \cup [0,1] \\
&= [0,2],
\end{aligned}
$$

while, for $x \in (1,2]$,

$$
\begin{aligned}
S(x) &= \{u \in [0,2] \mid s_x(u) > \alpha(x)\} = \{u \in [0,2] \mid s_x(u) > \frac{1}{2}\} \\
&= \{u \in (1,2] \mid 1 > \frac{1}{2}\} \cup \{u \in [0,1] \mid \frac{u+3}{4} > \frac{1}{2}\} \\
&= (1,2] \cup [0,1] \\
&= [0,2].
\end{aligned}
$$

Because of these reasons, $S(x) = [0,2]$ is nonempty and convex, for all $x \in K$, the set fix $S = [0,2]$ is closed, and $\overline{\cup_{x \in K} S(x)} = K$ is compact. Now, for any $x \in K$,

$$
\begin{aligned}
T(x) &= \{d \in [0,2] \mid t_x(d) > \beta(x)\} = \{d \in [0,2] \mid t_x(d) > \frac{1}{7+x^2}\} \\
&= \{d \in [0,2] \mid \frac{1}{1+2d} > \frac{1}{7+x^2}\} \\
&= [0,2],
\end{aligned}
$$

which is nonempty and convex. For all $(x,y,u) \in K \times D \times K$, we obtain that

$$
\begin{aligned}
G(x,y,u) &= \{z \in \mathbb{R} \mid g_{x,y,u}(z) > \gamma(x)\} = \{z \in \mathbb{R} \mid g_{x,y,u}(z) > \frac{1}{6+2x^2}\} \\
&= \{z \in [0,10] \mid \frac{1}{2+2z+\frac{y^2}{2}} > \frac{1}{6+2x^2}\} \cup \{z \in \mathbb{R} \setminus [0,10] \mid \frac{1}{15} > \frac{1}{6+2x^2}\} \\
&= [0, 2+x^2-\frac{y^2}{4}) \cup \emptyset = [0, 2+x^2-\frac{y^2}{4}).
\end{aligned}
$$

and, by the upper C-convexity of G in its third argument, for all $(x,y) \in K \times D$, we obtain the convexity of the set

$$\{u \in K \mid G(x,y,u) \not\subseteq Y \setminus (-\mathrm{int}\, C)\}.$$

It is easy to check that $G(x,y,x) \subseteq Y \setminus (-\mathrm{int}\, C)$, for all $x \in$ fix S and $y \in D$. The lower semicontinuity of the functions $x \mapsto s_x(u)$, $x \mapsto t_x(d)$, $x \mapsto g_{x,y,u}(z)$, $y \mapsto g_{x,y,u}(z)$, together with the upper semicontinuity of α, β, and γ completes the assumptions of Theorem 9, and any $(\bar{x}, \bar{y})$, with $\bar{x} \in [0,2]$ and $\bar{y} \in [0,2]$, is the solution to the vector quasi-equilibrium problem with fuzzy mappings.

6. Conclusions

The Browder-type fixed-point theorem, used throughout this paper, opens up a new research direction in the theory of vector quasi-equilibrium problems with set-valued mappings. Our results deal with some milder semicontinuity assumptions, but with different generalized convexity assumptions than those existent in the literature. On the moving cone C, we demand only that it is solid and convex, and no upper semicontinuity or closedness assumptions are made, as they are completed in the before-established results in this field.

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