mathematics

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Periodic Solutions to Nonlinear Second-Order Difference Equations with Two-Dimensional Kernel

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Abstract: In this work, we provide conditions for the existence of periodic solutions to nonlinear, second-order difference equations of the form $y(t+2)+by(t+1)+cy(t)=g(y(t))$, where b and c are real parameters, $c \neq 0$, and $g:\mathbb{R}\to\mathbb{R}$ is continuous.

Keywords: periodic difference equations; resonance; Lyapunov-Schmidt procedure; Schaefer's fixed point theorem

MSC: 39A23; 39A27

1. Introduction

In this work, we provide conditions for the existence of periodic solutions to nonlinear, second-order difference equations of the form

$$y(t+2)+by(t+1)+cy(t)=g(y(t)). \tag{1}$$

Throughout our discussion, we will assume that b and c are real parameters, $c \neq 0$, and $g:\mathbb{R}\to\mathbb{R}$ is continuous.

In the paper [1], the authors prove the existence of N-periodic solutions to (1) under various restrictions on the nonlinearity g, the parameters b and c, and the period N. Two of the most prominent results are the following:

Proposition 1. *Suppose that the following conditions hold:*

A1. $\lim\limits_{r\to\infty}\dfrac{\|g\|_r}{r}=0$, *where, for* $s>0$, $\|g\|_s=\sup\limits_{x\in[-s,s]}|g(x)|$;

A2. *there exists a positive number* $\hat{z}$ *such that* $xg(x)>0$ *whenever* $|x|>\hat{z}$;

A3. *if* $N\arccos(-\frac{b}{2})$ *is a multiple of* 2π, *then* $c\neq 1$ *or* $2\leq|b|$.

If N *is odd with* $N>1$, *then (1) has a* N*-periodic solution.*

Proposition 2. *Suppose the following conditions hold:*

B1. $c=1$, $|b|<2$, *and* $N\arccos(-\frac{b}{2})=2\pi r$ *for some* $r\in\mathbb{N}$;

B2. *the function* g *is bounded, say by* K;

B3. *there are constants* $\hat{z}$ *and* $J>0$ *such that for all* $x\in\mathbb{R}$ *with* $x\geq\hat{z}$, $g(-x)\leq -J<0<J\leq g(x)$;

B4. $\dfrac{N}{gcd(r,N)}\geq\max\left\{3,\dfrac{K}{J}+1\right\}$, *where* $gcd(r,N)$ *denotes the greatest common divisor of* r *and* N.

If N *is odd, then (1) has a* N*-periodic solution.*

Clearly, the assumptions of Proposition 1 generate the existence of solutions to (1) for a more general class of nonlinearities, g, than do the assumptions of Proposition 2, since unbounded nonlinearities can easily satisfy the conditions of Proposition 1. For particular

Citation: Maroncelli, D. Periodic Solutions to Nonlinear Second-Order Difference Equations with Two-Dimensional Kernel. *Mathematics* **2024**, *12*, 849. https://doi.org/10.3390/math12060849

Academic Editors: Lingju Kong and Min Wang

Received: 19 February 2024
Revised: 9 March 2024
Accepted: 12 March 2024
Published: 14 March 2024

examples of such g, see [1]. Now, the reason that Proposition 2 requires stronger conditions on the nonlinearity, g, is simple. In Proposition 1, the assumption A3. ensures that the dimension of the solution space to the N-periodic homogeneous problem

$$y(t+2) + by(t+1) + cy(t) = 0 \tag{2}$$

is one-dimensional. In Proposition 2, condition B1. forces the solution space to (2) to be two-dimensional. See the appendix for the details. When the solution space of (2) is two-dimensional, the analysis of (1) is more complex, as the interaction of the solution space and the nonlinearity is much more complicated. For this reason, additional requirements were placed.

As a final remark at the end of [1], the authors left open the question of whether similar results to Proposition 2 hold without a boundedness assumption placed on g. In particular, they posed the question of whether the existence of solutions to (1) could be proved when condition B1. holds, but under assumptions "similar" to A1. and A2. In this paper, we show that this is indeed the case; that is, we prove the existence of solutions to (1) when B1. holds and assumptions A1. and A2. are valid. Interestingly, we will also show that this more general result holds when N is even, something that is not discussed in [1], where they always assume N is odd. We will also discuss the existence of solutions to (1) when $b = 0$, $c = -1$, and N is even with $N \geq 4$. As it turns out, see the appendix for the details, for real parameters b and c, the case when $b = 0$ and $c = -1$ is the only case in which condition B1. does not hold, and the solution space of (2) is two-dimensional. So, in this regard, this paper shows that conditions A1. and A2. of Proposition 1 are sufficient to prove the existence of solutions to (1) in all cases where the solution space of (2) is two-dimensional.

To provide a bit more concreteness to the discussion above, we list here, for reference, our main result, Theorem 2, which will be proved in Section 3.

Theorem (Theorem 2)**.** *Suppose the following conditions hold:*

C1. *the solution space to* (2) *is two-dimensional;*

C2. $\lim_{s\to\infty} \frac{\|g\|_s}{s} = 0$, *where, for* $w > 0$, $\|g\|_w = \sup_{x\in[-w,w]} |g(x)|$;

C3. *there is a positive number* $\hat{z}$ *such that* $xg(x) > 0$ *whenever* $|x| > \hat{z}$.

Then (12) *has a* N*-periodic solution.*

Remark 1. *We would like to point out, while Propositions 1 and 2, and Theorem 2 are close by, that Theorem 2 is obviously a substantial generalization of Proposition 2; we will discuss the various "advantages" of Theorem 2 in more detail after the proof of Theorem 2. Additionally, Theorem 2 is also the "ideal" analog of Proposition 1 in the more complicated setting where the solution space to* (2) *is two-dimensional. However, as similar as the statements of Proposition 1 and Theorem 2 may be, their proofs take an entirely different route. In fact, the proof of Theorem 2 differs, almost in its entirety, from the original proofs of Propositions 1 and 2 (found in [1]), and it is this new approach that makes our work novel.*

The theory of periodic solutions to nonlinear differential/difference equations is extensive. Most of the deep results in this setting are for problems in which an associated linear homogeneous problem has at most a one-dimensional kernel. There are also some known results when the dimension of this solution space is odd but of a higher dimension. Very little is known in cases of resonance where the dimension of resonance is even. For those readers interested in known results in this area of study, we mention a few that are relevant to this work. In [2–5], periodic solutions are analyzed. In [6–11] the authors study the existence of solutions to nonlinear discrete Sturm-Liouville problems. Refs. [12–15] establish existence results for multi-point problems. Positive solutions are treated in [16–18]. Results regarding the existence of multiple solutions may be found in [19–21].

The paper is organized as follows: In Section 2, we introduce the preliminary ideas needed to study (1) from an operator theoretic point of view. Section 2 contains nothing novel and is included simply for completeness. Those familiar with the theory of linear difference equations at resonance can safely skim Section 2 and move directly to Section 3. Section 3 contains our main result, which is proved using Schaefer's fixed theorem. Section 4 contains an example showing the type of nonlinearities we had in mind when developing the main result, Theorem 2. Section 5 contains some concluding remarks. Lastly, in Appendix A, we conclude the paper with an appendix that contains calculations verifying the dimension of the solution space to (2) under various conditions on the real parameters b and c. The calculations in the appendix are not difficult; however, they are a bit tedious, which is why we have designated them as an appendix.

2. Preliminaries

We begin with several preliminary ideas that will be needed to develop our main result, Theorem 2. All of the statements in this section are well-known and can be found in [1]. We include these results to improve readability, especially for those who may not be experts in this area, and to make the document essentially self-contained.

Our approach to analyzing the nonlinear boundary value problem, (1), will be to view it as an operator problem for an equivalent system of difference equations. We start by defining

$$A = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix},$$

and $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(u, v) = \begin{pmatrix} 0 \\ g(u) \end{pmatrix}.$$

If we let $x(t)$ denote $\begin{pmatrix} y(t) \\ y(t+1) \end{pmatrix}$, then finding N-periodic solutions to (1) is equivalent to solving

$$x(t+1) = Ax(t) + f(x(t)) \tag{3}$$

subject to

$$x(0) - x(N) = 0. \tag{4}$$

To view our new system in an operator theoretic framework, we introduce the following function space and associated operators: First, we let

$$X_N = \left\{ \varphi : \mathbb{N}_0 \to \mathbb{R}^2 | \varphi \text{ is } N\text{-periodic} \right\}.$$

We view X_N as a finite-dimensional normed space using the supremum norm, which we will denote by $\|\cdot\|$. When needed, we will use $|\cdot|$ to denote the standard Euclidean norm on $\mathbb{R}^2$. We now define operators

$\mathcal{L}\colon X_N \to X_N$ by

$$(\mathcal{L}x)(t) = x(t+1) - Ax(t),$$

and

$\mathcal{F}\colon X_N \to X_N$ by

$$\mathcal{F}(x)(t) = f(x(t)).$$

It should be clear that finding N-periodic solutions to (1) is now equivalent to solving

$$\mathcal{L}x = \mathcal{F}(x). \tag{5}$$

As a first step in our analysis of the nonlinear boundary value problem (1), we analyze the linear nonhomogeneous problem $\mathcal{L}x = h$, where h is a N-periodic function. Our characterization of the $\text{im}(\mathcal{L})$ (the image of $\mathcal{L}$) will then be used to create a projection scheme, often referred to as the Lyapunov-Schmidt projection scheme, which will be used to analyze (1). The characterization of the $\mathrm{im}(\mathcal{L})$ is straightforward; it depends to a large extent on the fact that the principal fundamental matrix solution to

$$x(t+1) = Ax(t) \tag{6}$$

is given by $\Phi(t) = A^t$, where t is as in (6). For those readers not familiar with this result, we suggest [22,23]. Ref. [23] is a great resource for those already familiar with many standard results from the theory of linear ordinary differential equations. Ref. [22] has a nice introduction to several standard topics in difference equations, their discussion of periodic linear systems being the one most relevant to the work of this paper.

As our first introductory result, we completely characterize the $\mathrm{im}(\mathcal{L})$. As is often the case for differential and difference operators, the image of our mapping is "essentially" an orthogonal complement. As a matter of notation, since it will appear several times moving forward, we point out, that for any matrix C, we will use C^T to denote its transpose.

Proposition 3. *An element $h \in X_N$ is contained in the* $\mathrm{im}(\mathcal{L})$ *if and only if*

$$A^N \sum_{i=0}^{N-1} A^{-(i+1)} h(i) \in \ker\left(\left(I - A^N\right)^T\right)^{\perp},$$

where for any subspace E of $\mathbb{R}^2$, $E^{\perp} = \{v \in \mathbb{R}^2 \mid v^T w = 0 \text{ for all } w \in E\}$.

Proof. Suppose $\mathcal{L}x = h$ for some $x \in X_N$. Using the variation of parameters formula, we have

$$x(t) = A^t x(0) + A^t \sum_{i=0}^{t-1} A^{-(i+1)} h(i).$$

Since $x(0) = x(N)$, we must have that

$$x(0) = x(N) = A^N x(0) + A^N \sum_{i=0}^{N-1} A^{-(i+1)} h(i).$$

It now easily follows that $\mathcal{L}x = h$ if and only if $A^N \sum_{i=0}^{N-1} A^{-(i+1)} h(i) \in \mathrm{im}(I - A^N)$. The statement of the proposition is now a consequence of the fact that for any square matrix C, $\mathrm{im}(C) = \ker(C^T)^{\perp}$. □

We also have the following result regarding the linear homogeneous system, (6).

Corollary 1. *The* $\ker(\mathcal{L})$ *and* $\ker\left(I - A^N\right)$ *have the same dimension.*

Proof. From the proof of Proposition 3, $\mathcal{L}x = 0$ if and only if $x(t) = A^t v$ and $(I - A^N)v = 0$ for some $v \in \mathbb{R}^2$. □

Let W denote any matrix whose columns form a basis for $\ker\left((I - A^N)^T\right)$. It follows from Proposition 3 that $h \in \mathrm{im}(\mathcal{L})$ if and only

$$W^T A^N \sum_{i=0}^{N-1} A^{-(i+1)} h(i) = 0.$$

For $t \in \mathbb{N}_0$, we define $\Psi(t) = \begin{cases} (A^N)^T W & t = 0 \\ (A^{-(i+1)})^T (A^N)^T W & t > 0 \end{cases}$. It is then a routine verification to show that $\mathcal{L}x = h$ if and only if

$$\sum_{i=0}^{N-1} \Psi^T(i+1)h(i) = 0. \tag{7}$$

During the proof of Theorem 2, we will take advantage of the fact that the columns of Ψ span the solution space of the N-periodic linear homogeneous "adjoint" problem.

$$\mathcal{L}^* x = 0, \tag{8}$$

where $\mathcal{L}^* : X_N \to X_N$ is defined by

$$(\mathcal{L}^* x)(t) = x(t+1) - A^{-T}x(t).$$

As a reminder, $(\cdot)^T$ denotes transpose. If you know a bit about adjoint operators, $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. From the basic theory of linear difference equations, we have that any fundamental matrix solution to the "adjoint" problem, (8), is of the form $\Psi(t)D$, for some invertible matrix D. Using (7), we have that $\mathcal{L}x = h$ if and only if

$$\sum_{i=0}^{N-1} \Gamma^T(i+1)h(i) = 0 \tag{9}$$

for any fundamental matrix solution to (8), Γ.

We intend to prove the existence of solutions to (1) using a Schaefer fixed point argument. In this setting, it will be useful to know that the "adjoint" system produces periodic solutions to a scalar difference equation which is very similar to (1). In fact, in the cases of interest to this paper, the adjoint scalar difference equation and (1) agree. The derivation, regardless of the dimension of $\ker(\mathcal{L})$, proceeds along the following lines: Calculating A^{-T}, we get

$$A^{-T} = \frac{1}{c}\begin{pmatrix} -b & c \\ -1 & 0 \end{pmatrix}. \tag{10}$$

It is now easy to see that solving (8) is equivalent to

$$\begin{aligned} cx_1(t+1) &= -bx_1(t) + cx_2(t) \\ cx_2(t+1) &= -x_1(t) \end{aligned}$$

or

$$cx_2(t+2) + bx_2(t+1) + x_2(t) = 0.$$

Thus, the second component of a solution to the "adjoint" system is a N-periodic solution to

$$cy(t+2) + by(t+1) + y(t) = 0. \tag{11}$$

As was mentioned above, we intend to analyze the nonlinear periodic problem (1) using an alternative method in conjunction with Schaefer's fixed point theorem. Crucial to the use of this alternative method is the construction of projections onto the kernel and image of $\mathcal{L}$. The proofs of the following two results are trivial, so they are omitted. For readers interested in the proofs of Propositions 4 and 5, see [4].

Proposition 4. *Let V be the orthogonal projection onto* $\ker(I - A^N)$*. If we define $P : X_N \to X_N$ by $(Px)(t) = A^t V x(0)$, then P is a projection onto the* $\ker(\mathcal{L})$*.*

Proposition 5. *If we define* $Q : X_N \to X_N$ *by*

$$(Qh)(t) = \Psi(t)\left(\sum_{j=0}^{N-1} |\Psi(j)|^2\right)^{-1} \sum_{i=0}^{N-1} \Psi^T(i)h(i),$$

then Q *is a projection with* $\ker(Q) = \operatorname{im}(\mathcal{L})$.

The following is a formulation of the alternative method we will use to analyze (1). Since under our assumptions, $\ker(\mathcal{L})$ will be two-dimensional, $\mathcal{L}$ will not be invertible. When $\mathcal{L}$ is not invertible, using fixed point methods to analyze (1) is not straightforward. However, the development of the Lyapunov-Schmidt projection scheme will allow us to define a mapping, say H, on appropriate sequence spaces, for which the solutions to (1) are precisely the fixed points of H. For those readers interested in a more thorough treatment of alternative methods, we suggest [24]. Again, this result is well-known, we include the proof of this result for the benefit of the reader.

Proposition 6. *Solving* $\mathcal{L}x = \mathcal{F}(x)$ *is equivalent to solving the system*

$$\begin{cases} x = Px + M_p(I - Q)\mathcal{F}(x) \\ \text{and} \\ Q\mathcal{F}(x) = 0 \end{cases},$$

where M_p *is* $(\mathcal{L}_{|Ker(P)})^{-1}$.

Proof. $\mathcal{L}x = \mathcal{F}(x)$ for some $x \in X_N$ if and only if

$$\begin{cases} (I - Q)(\mathcal{L}x - \mathcal{F}(x)) = 0 \\ \text{and} \\ Q(\mathcal{L}x - \mathcal{F}(x)) = 0 \end{cases}.$$

Since $Q\mathcal{L}x = 0$, we conclude

$$\begin{cases} \mathcal{L}x - (I - Q)\mathcal{F}(x) = 0 \\ \text{and} \\ Q\mathcal{F}(x) = 0 \end{cases}.$$

Applying M_p to the first equation in the system gives

$$\begin{cases} M_p\mathcal{L}x - M_p(I - Q)\mathcal{F}(x) = 0 \\ \text{and} \\ Q\mathcal{F}(x) = 0 \end{cases},$$

which is equivalent to

$$\begin{cases} (I - P)x - M_p(I - Q)\mathcal{F}(x) = 0 \\ \text{and} \\ Q\mathcal{F}(x) = 0 \end{cases}.$$

□

Remark 2. *Since* $\ker(Q) = \operatorname{im}(\mathcal{L})$, $Q\mathcal{F}(x) = 0$ *if and only if*

$$\sum_{i=0}^{N-1} \Gamma^T(i+1)\begin{pmatrix} 0 \\ g(x(i)) \end{pmatrix} = 0$$

for all fundamental matrix solutions Γ to (8). We will return to this idea shortly when constructing the mapping, H, mentioned above.

3. Existence Results When $\dim(\ker(\mathcal{L})) = 2$

In this section, we prove our main existence theorem for the periodic difference Equation (1). As a reminder, we are interested in finding N-periodic solutions to

$$y(t+2) + by(t+1) + cy(t) = g(y(t)) \tag{12}$$

for $N \in \mathbb{N}$ with $N \geq 3$. Our interest will be limited to cases where the solution space is a linear, homogeneous problem

$$y(t+2) + by(t+1) + cy(t) = 0 \tag{13}$$

is two-dimensional, since in this case very little is known. As has been mentioned in the introduction and is proved in the appendix, the solution space to (13) is two-dimensional only in the following cases:

R1. $c = 1$, $|b| < 2$, and $N \arccos(-\frac{b}{2}) = 2\pi r$ for some $r \in \mathbb{N}$;
R2. $c = -1$, $b = 0$, and $N \in 2\mathbb{Z}$ with $N \geq 4$.

The analysis of (12) depends, to some extent, on which condition R1. or R2. holds, and so for the ease of the reader, we have broken the proof of our main result, Theorem 2, into two cases.

As has been mentioned in our earlier discussion of Theorem 2, we will prove the existence of solutions to (1) when B1. of Proposition 2 holds and assumptions A1. and A2. of Proposition 1 are valid. Existence will be proved using Schaefer's fixed point theorem, which we now state for the convenience of the reader.

Theorem 1 (Schaefer's Theorem)**.** *Let X be a finite-dimensional Banach space, and for $\nu > 0$, let $\overline{B}(0,\nu)$ denote the closed ball of radius ν centered at the origin, with $\partial\overline{B}(0,\nu)$ denoting its boundary. Suppose $T : X \to X$ is a continuous mapping. If there exists an $R > 0$ such that $S = \{(x,\lambda) \in \partial\overline{B}(0,R) \times (0,1) \mid x = \lambda T(x)\} = \varnothing$, then T has a fixed point in $\overline{B}(0,R)$.*

We now come to our main result.

Theorem 2. *Suppose the following conditions hold:*

C1. the solution space to (2) is two-dimensional, that is, suppose either R1. or R2. holds;

C2. $\displaystyle\lim_{s\to\infty} \frac{\|g\|_s}{s} = 0$, where, for $w > 0$, $\|g\|_w = \displaystyle\sup_{x\in[-w,w]} |g(x)|$;

C3. There exists a positive number $\hat{z}$ such that $xg(x) > 0$ whenever $|x| > \hat{z}$.

Then (12) has a N-periodic solution.

Proof. *(The case R1.)* We start by assuming that condition R1. holds; that is, we will be assuming that $c = 1$, $|b| < 2$, N is a fixed natural number with $N \geq 3$, and $N\theta = 2\pi r$ for some natural number r, where $\theta = \arccos\left(-\frac{b}{2}\right)$. In this case, see the appendix, it follows that

$$\Phi(t) = \begin{pmatrix} \cos(\theta t) & \sin(\theta t) \\ \cos(\theta(t+1)) & \sin(\theta(t+1)) \end{pmatrix}$$

is a fundamental matrix solution to (6). Since $c = 1$, we have found that the periodic scalar problems (1) and (11) agree, so that

$$\Gamma(t) = \begin{pmatrix} -\cos(\theta t) & -\sin(\theta t) \\ \cos(\theta(t-1)) & \sin(\theta(t-1)) \end{pmatrix}$$

is a fundamental matrix solution to the adjoint system (8).

Let

$$H(\alpha, x) = \begin{pmatrix} \alpha - \sum_{j=0}^{N-1} e^{i\theta j} g(\langle \alpha, e^{i\theta j} \rangle + [x]_1(j)) \\ M_p(I-Q)\mathcal{F}(\Phi(\cdot)\alpha + x) \end{pmatrix},$$

whenever $\alpha \in \mathbb{R}^2$ and $x \in \mathrm{im}(I-P)$, where here $e^{i\theta j} = \begin{pmatrix} \cos(\theta j) \\ \sin(\theta j) \end{pmatrix}$. From Proposition 6, Remark 2, and the discussion above, it follows that the solutions to (12) are precisely the fixed points of H. We will show that H has a fixed point using Schaefer's fixed point theorem.

The norm generating the topology on $\mathbb{R}^2 \times \mathrm{im}(I-P)$ is not terribly important, but for concreteness we make $\mathbb{R}^2 \times \mathrm{im}(I-P)$ a Banach space under the topology generated by the norm

$$\|(\alpha, x)\| = \max\{|\alpha|, \|x\|\}.$$

Let $S = \{(\alpha, x) \in \mathbb{R}^2 \times \mathrm{im}(I-P) \mid (\alpha, x) = \lambda H(\alpha, x) \text{ for some } \lambda \in (0,1)\}$. We will show that S is a bounded set, and thus, by Schaefer's theorem, H will have a fixed point. To reach a contradiction, suppose that S is unbounded and choose sequences $(\alpha_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}, (\lambda_n)_{n\in\mathbb{N}}$ with $(\alpha_n, x_n) = \lambda_n H(\alpha_n, x_n)$, and $\|(\alpha_n, x_n)\| \to \infty$. By going to subsequences if needed, we may assume that there exist $\alpha_0 \in \mathbb{R}^2, x_0 \in \mathrm{im}(I-P)$, and $\lambda_0 \in [0,1]$, with $\frac{1}{\|(\alpha_n, x_n)\|}(\alpha_n, x_n) \to (\alpha_0, x_0)$ and $\lambda_n \to \lambda_0$.

To simplify notation, for $\alpha \in \mathbb{R}^2$ and $x \in \mathrm{im}(I-P)$, let

$$p(\alpha, x) = M_p(I-Q)\mathcal{F}(\Phi(\cdot)\alpha + x).$$

Observe that any $\alpha \in \mathbb{R}^2$ and any $x \in \mathrm{im}(I-P)$

$$|\langle \alpha, e^{i\theta k} \rangle + [x]_1(k)| \le |\alpha| + |[x]_1(k)| \le |\alpha| + \|x\| \le 2\|(\alpha, x)\|,$$

where $[x]_1$ is the first component of the vector x. Therefore,

$$\begin{aligned} \|p(\alpha, x)\| &= \|M_p(I-Q)\mathcal{F}(\Phi(\cdot)\alpha + x)\| \\ &\le \|M_p(I-Q)\| \|\mathcal{F}(\Phi(\cdot)\alpha + x)\| \\ &= \|M_p(I-Q)\| \sup_{k\in\mathbb{N}_0} |g(\langle \alpha, e^{i\theta k} \rangle + [x]_1(k))| \\ &\le \|M_p(I-Q)\| \|g\|_{2\|(\alpha,x)\|}, \end{aligned} \tag{14}$$

where $\|M_p(I-Q)\| = \sup_{\|z\|=1} \|M_p(I-Q)z\|$.

From (14) and C2., we see that

$$\frac{p(\alpha_n, x_n)}{\|(\alpha_n, x_n)\|} \to 0.$$

Under essentially the same reasoning, we conclude that

$$\frac{H(\alpha_n, x_n)}{\|(\alpha_n, x_n)\|} \to (\alpha_0, 0).$$

But $(\alpha_n, x_n) = \lambda_n H(\alpha_n, x_n)$, so that

$$(\alpha_0, x_0) = \lim_{n\to\infty} \frac{1}{\|(\alpha_n, x_n)\|}(\alpha_n, x_n) = \lim_{n\to\infty} \frac{1}{\|(\alpha_n, x_n)\|}\lambda_n H(\alpha_n, x_n) = \lambda_0(\alpha_0, 0).$$

It follows that $\lambda_0 = 1$, $x_0 = 0$. Further, since $\|(\alpha_0, x_0)\| = 1$, we must have $|\alpha_0| = 1$.

Suppose for the moment that $\langle \alpha_0, e^{i\theta j}\rangle \neq 0$ for all $j \in \{0, \cdots, \frac{N}{gcd(r,N)} - 1\}$. Thus, $\frac{1}{\|(\alpha_n,x_n)\|}\langle \alpha_n, e^{i\theta j}\rangle \neq 0$ for all $j \in \{0, \cdots, \frac{N}{gcd(r,N)} - 1\}$ and large enough $n \in \mathbb{N}$. However, since $\frac{x_n}{\|(\alpha_n,x_n)\|} = \frac{\lambda_n p(\alpha_n,x_n)}{\|(\alpha_n,x_n)\|} \to 0$, we see that $\|(\alpha_n, x_n)\| = |\alpha_n|$ for large enough $n \in \mathbb{N}$. Since we are assuming that for every $j \in \{0, \cdots, \frac{N}{gcd(r,N)} - 1\}$ we have $\langle \alpha_0, e^{i\theta j}\rangle \neq 0$, it follows that $\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j) \to \pm\infty$ for all $j \in \{0, \cdots, \frac{N}{gcd(r,N)} - 1\}$ and that the sign (of $\pm\infty$) is that of $\langle \alpha_0, e^{i\theta j}\rangle$.

Since the collection $e^{i\theta j}, j = 0, \cdots, N-1$, is just $gcd(r, N)$ copies of the collection $e^{i\theta j}, j = 0, \cdots, \dfrac{N}{gcd(r,N)} - 1$, we easily deduce that $\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j) \to \pm\infty$ for each $j \in \{0, \cdots, N-1\}$ and that the sign is still the same as $\langle \alpha_0, e^{i\theta j}\rangle$. But then, for large enough $n \in \mathbb{N}$, we must have, using C3., that

$$\langle \alpha_n, e^{i\theta j}\rangle g(\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j)) > 0 \tag{15}$$

for all $j \in \mathbb{N}_0$, since the signs of $\langle \alpha_0, e^{i\theta j}\rangle$ and $\langle \alpha_n, e^{i\theta j}\rangle$ agree, at least for large enough $n \in \mathbb{N}$. It follows that for large enough $n \in \mathbb{N}$,

$$\begin{aligned}\left\langle \alpha_n, \sum_{j=0}^{N-1} e^{i\theta j} g(\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j)) \right\rangle &= \sum_{j=0}^{N-1} \langle \alpha_n, e^{i\theta j}\rangle g(\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j)) \\ &> 0.\end{aligned} \tag{16}$$

However, the result in (16) is contradictory, since from $(\alpha_n, x_n) = \lambda_n H(\alpha_n, x_n)$ we deduce

$$(1-\lambda_n)\alpha_n + \lambda_n \sum_{j=0}^{N-1} e^{i\theta j} g(\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j)) = 0, \tag{17}$$

so that by taking an inner product of the expression in (17) and α_n, we see that

$$(1-\lambda_n)|\alpha_n|^2 + \lambda_n \sum_{j=0}^{N-1} \langle \alpha_n, e^{i\theta j}\rangle g(\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j)) = 0, \tag{18}$$

which is not possible, since from (16), (18) is a sum of positive terms, at least for large $n \in \mathbb{N}$.

Our previous contradiction now forces $\langle \alpha_0, e^{i\theta j}\rangle = 0$ for some $j \in \{0, \cdots, \frac{N}{gcd(r,N)} - 1\}$. If we let $F = \{j \in \{0, \cdots, \frac{N}{gcd(r,N)} - 1\} \mid \langle \alpha_0, e^{i\theta j}\rangle \neq 0\}$, then

$$\begin{aligned}\sum_{j=0}^{N-1} \langle \alpha_0, e^{i\theta j}\rangle g(\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j)) &= gcd(r,N) \cdot \sum_{j\in F} \langle \alpha_0, e^{i\theta j}\rangle g(\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j)) \\ &> 0,\end{aligned}$$

whenever $F \neq \varnothing$, since as was just argued above, for all $j \in F$,

$$\langle \alpha_0, e^{i\theta j}\rangle g(\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j)) > 0,$$

whenever $n \in \mathbb{N}$ is large. Now it is entirely possible that $F = \varnothing$, but in this case, we trivially have

$$\sum_{j=0}^{N-1} \langle \alpha_0, e^{i\theta j}\rangle g(\langle \alpha_n, e^{i\theta j}\rangle + [x_n]_1(j)) = 0,$$

so that for all cases of N,

$$\sum_{j=0}^{N-1} \langle \alpha_0, e^{i\theta j} \rangle g(\langle \alpha_n, e^{i\theta j} \rangle + [x_n]_1(j)) \geq 0. \tag{19}$$

If we now have an inner product (17) with α_0, we deduce

$$(1 - \lambda_n)\langle \alpha_0, \alpha_n \rangle + \lambda_n \sum_{j=0}^{N-1} \langle \alpha_0, e^{i\theta j} \rangle g(\langle \alpha_n, e^{i\theta j} \rangle + [x_n]_1(j)) = 0. \tag{20}$$

However, (20) also produces a contradiction for large enough $n \in \mathbb{N}$. Indeed, since $|\alpha_n| \to \infty$ and $\langle \alpha_0, \frac{\alpha_n}{|\alpha_n|} \rangle \to \langle \alpha_0, \alpha_0 \rangle = 1$, we must have

$$\langle \alpha_0, \alpha_n \rangle = |\alpha_n| \langle \alpha_0, \frac{\alpha_n}{|\alpha_n|} \rangle \to \infty.$$

Further, by (19), $\sum_{j=0}^{N-1} \langle \alpha_0, e^{i\theta j} \rangle g(\langle \alpha_n, e^{i\theta j} \rangle + [x_n]_1(j)) \geq 0$ whenever $n \in \mathbb{N}$ is large. Thus, (20) must be positive for large enough $n \in \mathbb{N}$.

Since a contradiction is produced for all choices of α_0, it must be that S is bounded and so, by Schaefer's fixed point theorem, H has a fixed point. This fixed point is our solution to (12), which proves the existence of a solution to (12) in the case where condition R1. holds.

(The case R2.)

The proof for the case when condition R2. holds is very similar to what was given for the case when condition R1. is valid. Due to the similarity, we will not provide a complete proof for this case, but we do want to point out the few differences. First, in the case where R1. holds, see Appendix A, we have that

$$\Phi(t) = \begin{pmatrix} 1 & (-1)^t \\ 1 & -(-1)^t \end{pmatrix}$$

is a fundamental matrix solution to (6). However, since $b = 0$ and $c = -1$, the periodic scalar problems (1) and (11) once again agree. It follows that

$$\Gamma(t) = \begin{pmatrix} 1 & (-1)^t \\ 1 & -(-1)^t \end{pmatrix}$$

is a fundamental matrix solution to (8).

Let

$$H(\alpha, x) = \begin{pmatrix} \alpha - \sum_{j=0}^{N-1} \begin{pmatrix} 1 \\ (-1)^j \end{pmatrix} g(\langle \alpha, \begin{pmatrix} 1 \\ (-1)^j \end{pmatrix} \rangle + [x]_1(j)) \\ M_p(I - Q)\mathcal{F}(\Phi(\cdot)\alpha + x) \end{pmatrix},$$

whenever $\alpha \in \mathbb{R}^2$ and $x \in \operatorname{im}(I - P)$. Once again, it follows that the solutions to (12) are precisely the fixed points of H. The proof now proceeds, essentially as in the case when R1. holds, by assuming

$$S = \{(\alpha, x) \in \mathbb{R}^2 \times \operatorname{im}(I - P) \mid (\alpha, x) = \lambda H(\alpha, x) \text{ for some } \lambda \in (0, 1)\}$$

is unbounded and reaching a contradiction. The argument is almost identical; most of the changes consist of replacing $e^{i\theta j}$ by $\begin{pmatrix} 1 \\ (-1)^j \end{pmatrix}$ in the appropriate places. $\square$

Remark 3. *Theorem 2 is a substantial generalization of Proposition 2, since if g is bounded, then certainly assumption C2. is valid. Additionally, C3. is clearly satisfied when B3. of Proposition 2 is. It is also extremely important to note that condition B4. of Proposition 2 is no longer required.*

Remark 4. *In Proposition 2, it is assumed that N is odd. In Theorem 2, we make no such assumption. Thus, Theorem 2 not only generalizes Proposition 2 in that it allows for much more general nonlinearities, but it also generalizes it to allow for many more cases of the period N.*

4. Example

The simplicity of the hypotheses of Theorem 2 makes it very easy to visualize examples of nonlinearities, g, which will allow periodic solutions to (1). We now provide an example of a nonlinearity that we had in mind when formulating Theorem 2. Suppose either R1. or R2. holds, and let

$$g(x) = \ln(1 + |x|)\arctan(x) + \sin(x).$$

Clearly, g is continuous. It is obvious that for this choice of g, C2. holds, since, with our notation as in theorem 2, we have, for $s > 0$,

$$\|g\|_s \leq \ln(1 + s)\arctan(s) + 1.$$

It is also not hard to see that C3. holds. Thus, for this choice of g, (1) has a periodic solution under the conditions placed by either R1. or R2.

5. Concluding Remarks

We conclude our work, with the exception of the appendix, with a few closing remarks. First, even though it was not of interest to this paper, it is easy to establish that Proposition 1 can be extended from the assumption that N is odd to cases where N is even. This amounts to showing that $\ker(\mathcal{L})$ and $\ker(\mathcal{L}^*)$ have not changed in these cases where N is even. Lastly, there are several open questions in this setting that remain; I mention two that are of interest to the author. First, it is certainly of interest to know to what extent condition C2. of Theorem 2 can be weakened. Condition C2. is often referred to as a sublinear growth condition. It is currently an active area of research, in both nonlinear differential equations and nonlinear difference equations, to look for existence theorems under growth conditions on nonlinearities that are less stringent than sublinear growth. I encourage interested readers to look for existence results in this setting. Second, problem (1) is perfectly well-formulated when the parameters b and c are complex and the nonlinearity $g : \mathbb{C} \to \mathbb{C}$. The analysis in this complex setting is much more difficult, but it is certainly of interest to see to what extent Theorem 2 can be transferred to this complex setting.

Funding: This research received no external funding.

Data Availability Statement: No data sets were generated during this research.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A

In this final section, we present the characterizations of the $\ker(\mathcal{L})$ that were used in the proofs of our main result, Theorem 2. The calculations here are not difficult, but they do require a bit of tedious analysis, which is why they are deferred to this appendix.

As was shown in Proposition 1, $\mathcal{L}$ is singular if and only if $\ker(I - A^N)$ is singular, and in this case, $\dim(\ker(\mathcal{L}))$ is precisely equal to $\dim(\ker(I - A^N))$. In fact, in Proposition 1, we showed that if $\{v_1, \cdots, v_m\}$ is a basis for $\ker(I - A^N)$, then $\{\varphi_1, \cdots, \varphi_m\}$ is a basis of $\ker(\mathcal{L})$, where for $t \in \{0, \cdots, N\}$ and $k \in \{1, \cdots, m\}$, $\varphi_k(t) = A^t v_k$. Let us point out that since A is a 2×2 matrix, when $I - A^N$ is singular, we must have $m = 1$ or $m = 2$.

Now it is a simple characterization from linear algebra that $I - A^N$ is singular if and only if at least one eigenvalue of A is an Nth root of unity. However, since

$$A = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix},$$

We know that the eigenvalues of A are precisely the roots of the characteristic polynomial $p(z) = z^2 + bz + c$. In what follows, we show that:

D1. The dimension of the kernel of $\mathcal{L}$ is precisely the number of roots of the characteristic polynomial, which are Nth roots of unity.

The Case of a Repeated Root Is Considered to Have One Nth Root of Unity

Suppose that λ_1 and λ_2 are the complex roots of the characteristic polynomial $p(z) = z^2 + bz + c$. Since $(z - \lambda_1)(z - \lambda_2) = z^2 - (\lambda_1 + \lambda_2)z + \lambda_1\lambda_2$, we see that $b = -(\lambda_1 + \lambda_2)$ and $c = \lambda_1\lambda_2$. Note that since $c \neq 0$, neither λ_1 nor λ_2 is zero. If neither λ_1 nor λ_2 is a Nth root of unity, then from our discussion above, $\mathcal{L}$ is invertible, and so A1. holds in this case.

Now, without loss of generality, assume that λ_1 is an Nth root of unity. At the moment, suppose that $\lambda_1 \neq \lambda_2$. It is well-known that when $\lambda_1 \neq \lambda_2$,

$$\varphi_1(t) = \begin{pmatrix} \lambda_1^t \\ \lambda_1^{t+1} \end{pmatrix} \text{ and } \varphi_2(t) = \begin{pmatrix} \lambda_2^t \\ \lambda_2^{t+1} \end{pmatrix}$$

are linearly independent solutions to (6). If $c_1\varphi_1 + c_2\varphi_2$ was N-periodic, then we would have

$$\begin{pmatrix} c_1 + c_2 \\ c_1\lambda_1 + c_2\lambda_2 \end{pmatrix} = \begin{pmatrix} c_1\lambda_1^N + c_2\lambda_2^N \\ c_1\lambda_1^{N+1} + c_2\lambda_2^{N+1} \end{pmatrix} = \begin{pmatrix} c_1 + c_2\lambda_2^N \\ c_1\lambda_1 + c_2\lambda_2^{N+1} \end{pmatrix},$$

since λ_1 is an Nth root of unity. Equivalently, we have

$$\begin{pmatrix} c_2(1 - \lambda_2^N) \\ c_2\lambda_2(1 - \lambda_2^N) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If λ_2 is not an Nth root of unity, then $c_2 = 0$ and φ_1 must span $\ker(\mathcal{L})$. However, if λ_2 is an Nth root of unity, then φ_1, φ_2 must be a basis for $\ker(\mathcal{L})$. It follows that D1. holds for these cases of λ_1, λ_2.

The remaining case is when $\lambda_1 = \lambda_2$ and λ_1 is an Nth root of unity. As mentioned above, we are considering this case to have one Nth root of unity; D1. will be proved if we can show that $\dim(\ker(\mathcal{L})) = 1$ for this case. Now in the repeated roots case, it is well-known that

$$\varphi_1(t) = \begin{pmatrix} \lambda^t \\ \lambda^{t+1} \end{pmatrix} \text{ and } \varphi_2(t) = \begin{pmatrix} t\lambda^t \\ (t+1)\lambda^{t+1} \end{pmatrix}$$

are linearly independent solutions to (6), where $\lambda = \lambda_1 = \lambda_2$. If $c_1\varphi_1 + c_2\varphi_2$ was N-periodic, then we would have

$$\begin{pmatrix} c_1 \\ (c_1 + c_2)\lambda \end{pmatrix} = \begin{pmatrix} c_1 + Nc_2 \\ (c_1 + (N+1)c_2)\lambda \end{pmatrix},$$

since $\lambda^N = 1$. It follows easily that $c_2 = 0$ and that c_1 can be any complex constant; that is, φ_1 spans $\ker(\mathcal{L})$ and so $\dim(\ker(\mathcal{L})) = 1$.

From what was just shown, we know that $\dim(\ker(\mathcal{L})) = 2$ if and only if both roots of the characteristic polynomial $p(z) = z^2 + bz + c$ are roots of unity. We now look a bit more closely at these cases, under the assumptions that the coefficients b and c are real.

(Complex Roots)

If the parameters b and c are real, then in the case of complex roots, we must have that the roots are conjugate pairs. Thus, suppose that the roots of $z^2 + bz + c$ are λ and $\overline{\lambda}$, for some complex number λ. Here $\overline{\lambda}$ denotes the conjugate of λ. We then have that

$$z^2 + bz + c = (z - \lambda)(z - \overline{\lambda}) = z^2 - 2\,\mathrm{Re}(\lambda) + |\lambda|^2.$$

It follows that $b = -2\,\mathrm{Re}(\lambda)$ and $c = |\lambda|^2$.

Now λ is an Nth root of unity if and only if $\overline{\lambda}$ is an Nth root of unity, so if $\mathcal{L}$ (or equivalently $I - A^N$) is singular, then $|\lambda| = 1$. Since $c = |\lambda|^2$, we deduce that when $\mathcal{L}$ is singular, then $c = 1$. If we now write $\lambda = e^{i\theta}$ in polar form, then we also see that

$\mathrm{Re}(\lambda) = \cos(\theta)$ and so $b = -2\cos(\theta)$. Thus, in this complex setting, we have deduced the following: if $\mathcal{L}$ is singular, then $c = 1$ and $-2 < b < 2$. We point out that these conditions on b and c are necessary conditions for $\mathcal{L}$ to be singular, but they are certainly not sufficient.

In fact, we can say a bit more. If $\lambda = e^{i\theta}$ is an Nth root of unity, then we may arrange (swap λ and $\overline{\lambda}$ if needed) so that $\theta = \frac{2\pi r}{N}$ for some natural number r with $0 < r < \frac{N}{2}$. Rearranging gives $N\theta = 2\pi r$, where, from above, we would have $\theta = \arccos(-\frac{b}{2})$. It is well known that in this complex case,

$$\Phi(t) = \begin{pmatrix} \cos(\theta t) & \sin(\theta t) \\ \cos(\theta(t+1)) & \sin(\theta(t+1)) \end{pmatrix}$$

is a fundamental matrix solution to (6), as was claimed in the proof of Theorem 2.

(Real Distinct Roots)

The final case in which we may have that $\dim(\ker(\mathcal{L})) = 2$ is when the roots of the characteristic polynomial $p(z) = z^2 + bz + c$ are real and distinct. So, suppose that λ_1 and λ_2 are distinct roots of the characteristic polynomial $p(z) = z^2 + bz + c$. Since $(z - \lambda_1)(z - \lambda_2) = z^2 - (\lambda_1 + \lambda_2)z + \lambda_1\lambda_2$, we see that $b = -(\lambda_1 + \lambda_2)$ and $c = \lambda_1\lambda_2$. If λ_1 and λ_2 are both roots of unity, then we may assume $\lambda_1 = 1$ and $\lambda_2 = -1$. This forces N to be even. Our characteristic polynomial becomes $z^2 - 1$, so that $c = -1$ and $b = 0$. It is a simple consequence of the theory of linear difference equations that, in this case,

$$\Phi(t) = \begin{pmatrix} 1 & (-1)^t \\ 1 & (-1)^{t+1} \end{pmatrix}$$

is a fundamental matrix solution to (6), as was claimed in the proof of Theorem 2.

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