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Disturbing Fuzzy Multi-Attribute Decision-Making Method with If Weight Information Is Disturbing Fuzzy Number

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Abstract: Fuzzy multi-attribute decision-making is a hot research topic in which weight information is one of the conditions for forming a complete decision-making model, and it is also an important factor affecting the decision result. In most fuzzy multi-attribute decision-making problems, the weight information is often given in the form of real numbers. However, in real life, the weight information may not be suitable for specific numerical representation, or we cannot accurately determine the weight information. Therefore, it is very important to use fuzzy numbers to represent weight information. In this paper, we study the problem of disturbing fuzzy multi-attribute decision-making in which the attribute weight, decision-maker weight, and attribute information are given in the form of disturbing fuzzy numbers. Firstly, a new disturbing fuzzy integration operator, namely the disturbing fuzzy ring and multiplication aggregation (DFRMA) operator, is proposed, and its characteristics of closure, monotonicity, and boundary are studied. Then, the general steps of the disturbing fuzzy multi-attribute decision method based on the disturbing fuzzy ring and multiplication aggregation (DFRMA) operator are given, which include the single decision step and group decision step. Finally, an example is given to illustrate the practicability and effectiveness of the method.

Keywords: disturbing fuzzy sets; weight; multi-attribute decision-making; operator

MSC: 03E72; 91B06

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1. Introduction

Due to the uncertainty and complexity of the real world, this makes it impossible for classical mathematics to solve certain problems. In 1965, Zadeh [1] published an article called “Fuzzy Sets”, which marked the birth of fuzzy mathematics. The key idea of fuzzy mathematics is to acknowledge the “ambiguity” of intermediate transitions due to differences in objective things, allowing for the existence of “gradual relations”, that is, to admit that a set can have elements that belong partly to it and that a proposition can also be partly true and partly false. In today’s uncertain and complex society, this idea opens up a “new way” to solve a certain kind of problem. With the change in the world and the development of scientific theory, people find that the fuzzy sets established by Zadeh cannot be well applied to today’s world. Therefore, the value range of the membership function in general fuzzy set theory has been extended. Examples of these are interval-valued fuzzy sets [2], intuitive fuzzy sets [3], hesitant fuzzy sets [4], and so on. However, in the process of determining the degree of membership, there will be such a problem that when people describe an object, the person who is the subject of knowledge must first know the object. Due to differences in time, region, and personal cognition, it is impossible for people to recognize various objects without being affected by these factors [5]. For example, asking, “What is the membership of ’hot’ when the weather is 25 °C?”. Due to differences in time, region, and knowledge, people in city A think that their membership should be 0.80, people in city B may think that it is 0.73, and people in city C think that it is 0.66. It can be seen that people not only have a similar and general understanding of membership degree or the truth value of a fuzzy proposition but also have "interference" in the general value because of various objective or subjective factors. Therefore, Liu [5] extended the Zadeh fuzzy sets and proposed the disturbing fuzzy sets. It is not only better able to deal with the ambiguity of the real world but also more in line with the laws of human thinking.

At present, the research on disturbing fuzzy set theory is mainly concentrated in China. The two people who contributed most to this theory are Liu [5], the "mother" of disturbing fuzzy set theory, and Han [6]. Liu creatively incorporated the characteristic of the difference in the process of describing objective objects by the human brain into the general fuzzy set theory and created the disturbing fuzzy set. In addition to that, she introduced the concept of disturbing fuzzy propositional logic and defined the operators of disturbing fuzzy propositions; then she extended the 1-dimensional truth value of fuzzy logic operators to two-dimensional operators, which include disturbing fuzzy negation operators, implication operators, "and" and "or" operators and continuous operators, the properties of these logic operators are studied. These theories have laid a solid foundation for the future study of disturbing fuzzy sets. Han's main contribution was to combine the theory of disturbing fuzzy sets with other neighborhoods. Firstly, she compares the disturbing fuzzy set with the current mainstream interval-valued fuzzy set and intuitionistic fuzzy set and expounds on the advantages of the disturbing fuzzy set over the former two, that is, complete symmetry. This excellent property gives the disturbing fuzzy set theory an unprecedented advantage in tautologies and generalized tautologies. Secondly, she defines some algorithms and proves that the algebraic structure of disturbing fuzzy sets and their related operations is a superior soft algebra. She also gives the "bridge" between the disturbing fuzzy set and the ordinary set, namely the decomposition theorem and the representation theorem of the disturbing fuzzy set, which are the most basic theorems of the disturbing fuzzy set theory, and also clarifies the relationship between the ordinary set and the disturbing fuzzy set. These theories have laid a more solid foundation for the theory of disturbing fuzzy sets. Thirdly, on the basis of Liu, she conducted a more in-depth study of disturbing fuzzy logic. She introduced the implication operator in the disturbing fuzzy set and studied the algebraic structure and properties of the generalized tautologies of disturbing fuzzy and their mutual relations. Then, she introduces disturbing fuzzy semantics into the system L^* and studies disturbing fuzzy subgroups and disturbing fuzzy normal subgroups. The inference problem of disturbing fuzzy logic is studied, and the triple I algorithm of disturbing fuzzy inference is given. Fourthly, she combines the disturbing fuzzy set theory with the rough set theory for the first time and defines the rough, disturbing fuzzy set. The roughness of rough-disturbing fuzzy sets is given. By introducing the concept of horizontal upper and lower boundary of rough-disturbing fuzzy sets, the situation of the upper approximate intersection of two sets and the upper approximation of the intersection of two sets is not equal is solved. In addition, she defines the inclusion degree of a class of disturbing fuzzy set, studies some basic properties, and proves that the upper and lower approximation of rough, disturbing fuzzy set is a special example of the inclusion degree of disturbing fuzzy set and uses the proposed new knowledge to solve a specific operation problem of bank credit card application. Fifthly, she applied the theory of disturbing fuzzy set to the multi-attribute decision-making problem for the first time and proposed a new kind of fuzzy multi-attribute decision-making problem, namely the disturbing fuzzy multi-attribute decision-making problem. To solve this new problem, she proposed the possibility degree comparison method of disturbing fuzzy numbers and solved the problem that some disturbing fuzzy numbers cannot be compared by partial order relation (Definition 4). Inspired by the traditional TOPSIS method, three kinds of disturbing fuzzy multi-attribute decision-making methods based on the TOPSIS method are proposed; the six disturbing fuzzy integration operators are proposed, and their excellent properties, relations, and differences are studied. In addition, she proposed the disturbing fuzzy multi-attribute group decision-making method based on the disturbing fuzzy integration operator and possibility degree, etc. These methods can solve the disturbing fuzzy multi-attribute decision-making problem when the weights are completely known and are real numbers. She applied the proposed theory to the selection of tourist attractions, the carrying capacity evaluation of regional water resources, the evaluation of personnel management, and the selection of machine systems, which provided a new method for managers to make decisions.

In addition to the above two scholars who have contributed the most, there are other scholars who have also studied the disturbing fuzzy set theory. For example, Li [7] proved that the roughness measure of a disturbing fuzzy set is bounded and applied to the problem of grouping different students in a competition. Jiang [8] applies disturbing fuzzy set theory to R_0-algebra, gives the concept of disturbing fuzzy subalgebra and disturbing fuzzy MP filter, and discusses the equivalent characterization of disturbing fuzzy subalgebra on R_0-algebra and some properties of disturbing fuzzy MP filter; the homomorphic and inverse images of disturbing fuzzy MP filters are studied. Liu [9] introduced the concept of the join of disturbing-valued fuzzy finite-state machines, discussed the operations of disturbing-valued fuzzy sets on machines, and obtained some about the joining of disturbing-valued fuzzy finite-state machines. The contributions made by these scholars to the theory of disturbing fuzzy sets are irreplaceable, which has led to the gradual growth of disturbing fuzzy sets from a "baby" to an "adult". In addition, the theory of disturbing fuzzy sets and its combination with other theories are mainly studied in China, and one of the author's writing purposes is to make this new fuzzy set attract attention worldwide.

Fuzzy multi-attribute decision-making problems, such as interval-valued fuzzy multi-attribute decision-making [10], intuitionistic fuzzy multi-attribute decision-making [11], disturbing fuzzy multi-attribute decision-making [6], and hesitancy fuzzy multi-attribute decision-making [12], have been widely concerned because of their ability to better deal with uncertainty in decision-making. Fuzzy multi-attribute decision-making is mainly developed in utility theory [13] and hierarchical priority theory [14]. No matter what kind of theory, the aggregation operator plays an irreplaceable and supreme role. For example, Xu [15–17] studied some intuitionistic fuzzy aggregation operators, such as the intuitionistic fuzzy weighted geometric operator, the intuitionistic fuzzy ordered weighted geometric operator, the intuitionistic fuzzy ordered weighted averaging operator, and the intuitionistic fuzzy hybrid aggregation operator, for aggregating intuitionistic fuzzy values and establish various properties of these operators; Xia [18,19] studied some hesitant fuzzy aggregation operators, several series of aggregation operators are proposed, and the connections of them are discussed; Han [6] studied some disturbing fuzzy aggregation operators, such as disturbing fuzzy weighted arithmetic mean aggregation operator, disturbing fuzzy combination weighted geometric mean aggregation operator, disturbing fuzzy ordered weighted arithmetic mean aggregation operator, etc. The properties of these operators and the disturbing fuzzy multi-attribute decision-making methods based on these operators are studied. The birth of these operators makes the fuzzy multi-attribute decision theory develop as fast as "the speed of an airplane".

In addition, weight plays an important role in decision-making. Although sometimes people cannot give accurate weight information or only give partial weight information, we can still determine the exact value of the weight through the existing information. However, there is an extreme situation: the real world is full of ambiguity and uncertainty, and sometimes we cannot accurately determine the weight, or the weight may not be suitable for specific numerical representation. Lin [20] defines the weight represented by a fuzzy number as the "importance degree" of the attribute in the fuzzy sense. In this case, the fuzzy multi-attribute decision method represented by fuzzy numbers is extremely important. At present, only Lin has raised this kind of problem and provided a way to solve it. Other solutions have not been reported.

Therefore, based on the above discussion and the existing research shortcomings, this paper proposes a disturbing fuzzy multi-attribute decision-making method with if weight information is a disturbing fuzzy number. This paper proposes a new disturbing fuzzy aggregation operator and discusses and proves some properties of the operator, such as closure, monotonicity, and boundary. The detailed steps of a disturbing fuzzy multi-attribute decision-making method with if weight information is disturbing fuzzy number are given, including individual decision-making step and group decision-making step, so that the decision-maker can choose different decision steps according to different situations, so the method is more flexible. In addition, the method not only enriches the application range of disturbing fuzzy multi-attribute decision-making, but it also needs smaller calculations, and it can provide a useful way to efficiently help the decision-maker to make his decision. Furthermore, demonstrates the practicability and feasibility of the method with examples.

2. Preliminaries

In this section, we will discuss the concept of disturbing fuzzy sets, the relationship between fuzzy sets and disturbing fuzzy sets, and the similarities and differences between interval-valued fuzzy sets and disturbing fuzzy sets.

Definition 1 ([1])**.** *If X is a collection of objects denoted generically by x, then a fuzzy set $\widetilde{A}$ in X is a set of ordered pairs:*

$$\widetilde{A} = \{(x, \mu_A(x))|x \in X\} \tag{1}$$

$\mu_A(x)$ is called the membership function or grade of membership of x in $\widetilde{A}$, which maps X to the membership space M.

Definition 2 ([5])**.** *Let a set X be fixed; a disturbing fuzzy set Ω in X is given by as follows:*

$$\Omega = \{\langle x, A_\mu(x), A_\delta(x)\rangle | x \in X\} \tag{2}$$

where

$$A_\mu : X \to [0,1], x \in X \to A_\mu(x) \in [0,1] \tag{3}$$

$$A_\delta : X \to [0,1], x \in X \to A_\delta(x) \in [0,1] \tag{4}$$

$A_\mu(x)$ represents people's overall understanding of the object and $A_\delta(x)$ represents the "disturbing" caused by the "interference" of the understanding of the object caused by the subjective or objective difference of the individual. $A(x) = (A_\mu(x), A_\delta(x))$ changes in $[\max\{0, A_\mu(x) - A_\delta(x)\}, \min\{1, A_\mu(x) + A_\delta(x)\}]$.

Remark 1. *There are two differences between the disturbing fuzzy sets and the fuzzy sets. First, the membership function of the disturbing fuzzy sets does not take a single value inside but takes a pair of ordered arrays above. Second, from the definition of disturbing fuzzy sets, it can be seen that disturbing fuzzy sets are more consistent with the law of human thinking than fuzzy sets.*

Some scholars confuse the disturbing fuzzy sets with the interval-valued fuzzy sets when they first contact it. Here, we will analyze the two types of fuzzy sets.

Definition 3 ([2])**.** *Let a set X be fixed; an interval-valued fuzzy set $A_{I_{[0,1]}}$ in X is given by as follows:*

$$A_{I_{[0,1]}} = \{[x, A^-(x), A^+(x)] | x \in X\} \tag{5}$$

where

$$A^-(x) : X \to [0,1], A^+(x) : X \to [0,1], A^-(x) \le A^+(x) \tag{6}$$

$A^-(x)$ and $A^+(x)$ are called a lower fuzzy of $A_{I_{[0,1]}}$ and an upper fuzzy set of $A_{I_{[0,1]}}$, respectively.

Remark 2. *From the definition point of view, the logarithms of interval-valued fuzzy sets respectively indicate that people's understanding of objects changes in the interval formed by the upper and lower bounds of these two numbers. The logarithms of disturbing fuzzy sets respectively indicate* *that people's understanding of the object has a general "consensus", which is also affected by time, region, and personal opinions, and the "disturbing" above and below the "consensus" are respectively proposed by adapting to different human thinking angles, with both commonalities and differences. On the one hand, when $A^-(x) = A^+(x)$, the interval-valued fuzzy sets degenerate into a fuzzy set; when $\delta = 0$, the disturbing fuzzy set also degenerates into fuzzy sets so both of them are generalizations of fuzzy sets, which is their common point. On the other hand, each disturbing fuzzy number corresponds to an interval fuzzy number, such as $(0.6, 0.3)$ corresponding to $[0.3, 0.9]$. However, an interval fuzzy number can correspond to multiple disturbing fuzzy numbers (when $\mu \le \delta$), such as $[0, 0.6]$ corresponding to $(0.3, 0.3)$ and $(0.2, 0.4)$. Therefore, the relationship between interval-valued fuzzy sets and disturbing fuzzy sets is not a one-to-one correspondence, which is the difference between them.*

Definition 4 ([6])**.** *If $\alpha = (\alpha_\mu, \alpha_\delta)$ and $\beta = (\beta_\mu, \beta_\delta)$ are two disturbing fuzzy numbers, then the size relationship between them is defined as follows:*

$$\alpha \le \beta \Leftrightarrow \alpha_\mu \le \beta_\mu, \alpha_\delta \ge \beta_\delta \tag{7}$$

$$\alpha < \beta \Leftrightarrow \alpha_\mu < \beta_\mu, \alpha_\delta \ge \beta_\delta \quad or \quad \alpha_\delta \ge \beta_\delta, \alpha_\delta > \beta_\delta \tag{8}$$

$$\alpha = \beta \Leftrightarrow \alpha_\mu = \beta_\mu, \alpha_\delta = \beta_\delta \tag{9}$$

There are some disturbing fuzzy numbers whose magnitude cannot be compared by defining 4, such as $\alpha = (0.25, 0.53)$ and $\beta = (0.35, 0.63)$. Therefore, we propose a Boolean matrix ranking method for disturbing fuzzy numbers.

Definition 5 ([6])**.** *If $\alpha = (\alpha_\mu, \alpha_\delta)$ and $\beta = (\beta_\mu, \beta_\delta)$ are two disturbing fuzzy numbers,*

$$p(\alpha \ge \beta) = \min\left\{\max\left(\frac{\alpha_\mu + \alpha_\delta - \beta_\mu + \beta_\delta}{2(\alpha_\delta + \beta_\delta)}, 0\right), 1\right\} \tag{10}$$

then $p(\alpha \ge \beta)$ is called the probability of $\alpha \ge \beta$.

Theorem 1 ([6])**.** *If $\alpha = (\alpha_\mu, \alpha_\delta)$ and $\beta = (\beta_\mu, \beta_\delta)$ are two disturbing fuzzy numbers, and $p(\alpha \ge \beta)$ is the probabilityof $\alpha \ge \beta$, then*

(1) $0 \le p(\alpha \ge \beta) \le 1$;
(2) $p(\alpha \ge \beta) + p(\beta \ge \alpha) = 1$, in particular $p(\alpha \ge \alpha) = 1/2$.

Definition 6 ([6])**.** *If $\alpha_i = (\alpha_\mu^i, \alpha_\delta^i)$ is a set of disturbing fuzzy numbers, $1 \le i \le m$, denoted $p(\alpha_i \ge \alpha_j) = p_{ij}$, $1 \le i \le m$, $1 \le j \le m$, then $P = (p_{ij})_{m \times m}$ is called the possibility matrix.*

Definition 7. *If $P = (p_{ij})_{m \times m}$ is the possibility matrix, then $Q = (q_{ij})_{m \times m}$ is the Boolean matrix, where*

$$q_{ij} = \begin{cases} 1, p_{ij} \ge 0.5 \\ 0, p_{ij} < 0.5 \end{cases} \tag{11}$$

The specific steps of the Boolean matrix sorting method of disturbing fuzzy numbers:

Step 1: If $\alpha_i = (\alpha_\mu^i, \alpha_\delta^i)$ is a set of disturbing fuzzy numbers, $1 \le i \le m$, the possibility matrix is constructed by Formula (10);
Step 2: The Boolean matrix is constructed by the Formula (11);
Step 3: Let $\rho_i = \sum_{j=1}^m q_{ij}$ obtain the ranking vector $\rho = (\rho_1, \cdots, \rho_m)^T$;
Step 4: The disturbing fuzzy numbers are ranked according to the $\rho = (\rho_1, \cdots, \rho_m)^T$; the larger ρ_i, the larger α_i.

Example 1. *Let $\alpha_1 = (0.4, 0.4)$, $\alpha_2 = (0.69, 0.21)$, and $\alpha_3 = (0.72, 0.5)$. We find that α_1 and α_2 can be sized by defining 4, but α_1 and α_3, α_2 and α_3 cannot be sized by defining 4, so we use the Boolean matrix method of disturbing fuzzy numbers to rank the size of these three disturbing fuzzy numbers.*

From Formula (10), we can obtain:

$$p(\alpha_1 \geq \alpha_2) = 0.26,\ p(\alpha_1 \geq \alpha_3) = 0.32,\ p(\alpha_2 \geq \alpha_3) = 0.48;$$

Constructing the possibility matrix P:

$$P = \begin{bmatrix} 0.50 & 0.26 & 0.32 \\ 0.74 & 0.50 & 0.48 \\ 0.68 & 0.52 & 0.50 \end{bmatrix};$$

The Boolean matrix Q is constructed from Equation (11):

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix};$$

Obtaining the ranking vector $\rho = (1, 2, 3)^T$, so $\alpha_3 \succ \alpha_2 \succ \alpha_1$.

3. Disturbing Fuzzy Ring and Multiplication Aggregation Operator

In this section, we construct a new disturbing fuzzy aggregation operator, that is, the disturbing fuzzy ring and multiplication aggregation (DFRMA) operator, inspired by the ring operator and multiplication operator, and prove that the DFRMA operator has closure, monotonicity, and boundary.

Definition 8 ([21]). Let $\widetilde{A}$, $\widetilde{B}$ *be fuzzy set on* X *, and* $\forall x \in X$*, define:*
Ring operator$(\hat{+})$*:*

$$(A \hat{+} B)(x) = A(x) + B(x) - A(x)B(x);$$

Multiplication operator $(\bullet)$*:*

$$(A \bullet B)(x) = A(x)B(x).$$

If $S_p(x, y) = x + y - xy$, then $S_p(x, y)$ has the following properties [21]:

(1) Boundary: $S_p(1, 1) = 1$, $S_p(x, 0) = S_p(0, y) = 0$;
(2) Monotonicity: If $x \leq x_1$, $y \leq y_1$, then $S_p(x, y) \leq S_p(x_1, y_1)$;
(3) Replacement invariance: $S_p(x, y) = S_p(y, x)$;
(4) Associative law: $S_p(S_p(x, y), z) = S_p(x, S_p(y, z))$.

If $T_p(x, y) = xy$, then $T_p(x, y)$ has the following properties [21]:

(1) Boundary: $T_p(0, 0) = 0$, $T_p(y, 1) = 1$;
(2) Monotonicity: If $x \leq x_1$, $y \leq y_1$, then $T_p(x, y) \leq T_p(x_1, y_1)$;
(3) Replacement invariance: $T_p(x, y) = T_p(y, x)$;
(4) Associative law: $T_p(T_p(x, y), z) = T_p(x, T_p(y, z))$.

The ring operator and multiplication operator have some good properties. Therefore, in order to solve the disturbing fuzzy multi-attribute decision problem with the weight of disturbing fuzzy number, we construct a new operator.

Definition 9. *Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a set of disturbing fuzzy numbers, where $\alpha_i = (\mu_{\alpha_i}, \delta_{\alpha_i})$, $i = 1, 2, \cdots, n$; The disturbing fuzzy number $w_i = (\mu_{w_i}, \delta_{w_i})$ is the weight of α_i, and $w = (w_1, w_2, \cdots, w_n)$ is the weight vector, then*

$$\begin{aligned} DFRMA_w(\alpha_1, \alpha_2, \cdots, \alpha_n) &= \hat{+}_{i=1}^{n}(\alpha_i \bullet w_i) \\ &= (\hat{+}_{i=1}^{n}(\mu_{\alpha_i} \bullet \mu_{w_i}), \bullet_{i=1}^{n}(\delta_{\alpha_i} \hat{+} \delta_{w_i})) \end{aligned} \tag{12}$$

it is called the disturbing fuzzy ring and multiplication aggregation operator.

Proposition 1 (Closure). *Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a set of disturbing fuzzy numbers, where $\alpha_i = (\mu_{\alpha_i}, \delta_{\alpha_i})$, $i = 1, 2, \cdots, n$; the disturbing fuzzy number $w_i = (\mu_{w_i}, \delta_{w_i})$ is the weight of α_i, and $w = (w_1, w_2, \cdots, w_n)$ is the weight vector; then, $DFRMA_w(\alpha)$ is a disturbing fuzzy number.*

Proof. Because $\alpha_i = (\mu_{\alpha_i}, \delta_{\alpha_i})$, $w_i = (\mu_{w_i}, \delta_{w_i})$ are disturbing fuzzy numbers, so $\mu_{\alpha_i}, \delta_{\alpha_i}$ and $\mu_{w_i}, \delta_{w_i} \in [0, 1]$. Because $(\hat{+})$ and $(\bullet)$ are boundary, so $\delta_{\alpha_i} \hat{+} \delta_{w_i}$ and $\mu_{\alpha_i} \bullet \mu_{w_i} \in [0, 1]$, $i = 1, 2, \cdots, n$.

When $n = 1$, $\mu_{\alpha_1} \bullet \mu_{w_1} \in [0, 1]$;

When $n = 2$, $\mu_{\alpha_1} \bullet \mu_{w_1}$ and $\mu_{\alpha_2} \bullet \mu_{w_2} \in [0, 1]$, the boundary of $(\hat{+})$ tells us that $\hat{+}_{i=1}^{2}(\mu_{\alpha_i} \bullet \mu_{w_i}) \in [0, 1]$;

When $n = k$, there is $\hat{+}_{i=1}^{k}(\mu_{\alpha_i} \bullet \mu_{w_i}) \in [0, 1]$; then, when $n = k + 1$, there is

$$\hat{+}_{i=1}^{k+1}(\mu_{\alpha_i} \bullet \mu_{w_i}) = (\hat{+}_{i=1}^{k}(\mu_{\alpha_i} \bullet \mu_{w_i})) \hat{+} (\mu_{\alpha_{k+1}} \bullet \mu_{w_{k+1}}).$$

Because $\hat{+}_{i=1}^{k}(\mu_{\alpha_i} \bullet \mu_{w_i})$ and $(\mu_{\alpha_{k+1}} \bullet \mu_{w_{k+1}}) \in [0, 1]$, we know $\hat{+}_{i=1}^{k+1}(\mu_{\alpha_i} \bullet \mu_{w_i}) \in [0, 1]$ from $n = 2$, and then we know $\hat{+}_{i=1}^{n}(\mu_{\alpha_i} \bullet \mu_{w_i}) \in [0, 1]$ by mathematical induction;

When $n = 1$, $\delta_{\alpha_1} \hat{+} \delta_{w_1} \in [0, 1]$;

When $n = 2$, $\delta_{\alpha_1} \hat{+} \delta_{w_1}$ and $\delta_{\alpha_2} \hat{+} \delta_{w_2} \in [0, 1]$, the boundary of $(\bullet)$ tells us that $\bullet_{i=1}^{2}(\mu_{\alpha_i} \hat{+} \mu_{w_i}) \in [0, 1]$;

When $n = k$, there is $\bullet_{i=1}^{k}(\mu_{\alpha_i} \hat{+} \mu_{w_i}) \in [0, 1]$; then, when $n = k + 1$, there is

$$\bullet_{i=1}^{k+1}(\mu_{\alpha_i} \hat{+} \mu_{w_i}) = (\bullet_{i=1}^{k}(\mu_{\alpha_i} \hat{+} \mu_{w_i})) \bullet (\mu_{\alpha_{k+1}} \hat{+} \mu_{w_{k+1}}).$$

Because $\bullet_{i=1}^{k}(\mu_{\alpha_i} \hat{+} \mu_{w_i})$ and $(\mu_{\alpha_{k+1}} \hat{+} \mu_{w_{k+1}}) \in [0, 1]$, we know $\bullet_{i=1}^{k+1}(\mu_{\alpha_i} \hat{+} \mu_{w_i}) \in [0, 1]$ from $n = 2$, and then we know $\bullet_{i=1}^{n}(\mu_{\alpha_i} \hat{+} \mu_{w_i}) \in [0, 1]$ by mathematical induction. □

Proposition 2 (Monotonicity). *If $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ are two sets of different disturbing fuzzy numbers, where $\alpha_i = (\mu_{\alpha_i}, \delta_{\alpha_i})$, $\beta_i = (\mu_{\beta_i}, \delta_{\beta_i})$, and $\mu_{\alpha_i} \leq \mu_{\beta_i}$, $\delta_{\alpha_i} \geq \delta_{\beta_i}$, that is $\alpha_i \leq \beta_i$, $i = 1, 2, \cdots, n$; thedisturbing fuzzy number $w_i = (\mu_{w_i}, \delta_{w_i})$ is the common weight of α_i, β_i, $w = (w_1, w_2, \cdots, w_n)$ is the weight vector; then,*

$$DFRMA_w(\alpha) \leq DFRMA_w(\beta).$$

Proof. Because $\forall i$, $i = 1, 2, \cdots, n$, there is $\mu_{\alpha_i} \leq \mu_{\beta_i}$, $\delta_{\alpha_i} \geq \delta_{\beta_i}$, and by the monotonicity of $(\bullet)$ and $(\hat{+})$, then

$$\mu_{\alpha_i} \bullet \mu_{w_i} \leq \mu_{\beta_i} \bullet \mu_{w_i} \text{ and } \delta_{\alpha_i} \hat{+} \delta_{w_i} \geq \delta_{\beta_i} \hat{+} \delta_{w_i};$$

When $n = 1$,

$$\mu_{\alpha_1} \bullet \mu_{w_1} \leq \mu_{\beta_1} \bullet \mu_{w_1} \text{ and } \delta_{\alpha_1} \hat{+} \delta_{w_1} \geq \delta_{\beta_1} \hat{+} \delta_{w_1}$$

We can see from Definition 4 that $DFRMA_w(\alpha) \leq DFRMA_w(\beta)$;

When $n = 2$,

$$\mu_{\alpha_1} \bullet \mu_{w_1} \leq \mu_{\beta_1} \bullet \mu_{w_1} \text{ and } \delta_{\alpha_1} \hat{+} \delta_{w_1} \geq \delta_{\beta_1} \hat{+} \delta_{w_1},$$

$$\mu_{\alpha_2} \bullet \mu_{w_2} \leq \mu_{\beta_2} \bullet \mu_{w_2} \text{ and } \delta_{\alpha_2} \hat{+} \delta_{w_2} \geq \delta_{\beta_2} \hat{+} \delta_{w_2}$$

We know from the monotonicity of $(\bullet)$ and $(\hat{+})$:

$$\hat{+}_{i=1}^{2}(\mu_{\alpha_i}\bullet\mu_{w_i}) \leq \hat{+}_{i=1}^{2}(\mu_{\beta_i}\bullet\mu_{w_i}),$$

$$\bullet_{i=1}^{2}(\delta_{\alpha_i}\hat{+}\delta_{w_i}) \geq \bullet_{i=1}^{2}(\delta_{\beta_i}\hat{+}\delta_{w_i})$$

So

$$DFRMA_w(\alpha) = (\hat{+}_{i=1}^{2}(\mu_{\alpha_i}\bullet\mu_{w_i}), \bullet_{i=1}^{2}(\delta_{\alpha_i}\hat{+}\delta_{w_i})),$$

$$DFRMA_w(\beta) = (\hat{+}_{i=1}^{2}(\mu_{\beta_i}\bullet\mu_{w_i}), \bullet_{i=1}^{2}(\delta_{\beta_i}\hat{+}\delta_{w_i}))$$

We can see from Definition 4 that $DFRMA_w(\alpha) \leq DFRMA_w(\beta)$;
If when $n = k$ there is $DFRMA_w(\alpha) \leq DFRMA_w(\beta)$, then

$$\hat{+}_{i=1}^{n}(\mu_{\alpha_i}\bullet\mu_{w_i}) \leq \hat{+}_{i=1}^{n}(\mu_{\beta_i}\bullet\mu_{w_i}),$$

$$\bullet_{i=1}^{n}(\delta_{\alpha_i}\hat{+}\delta_{w_i}) \geq \bullet_{i=1}^{n}(\delta_{\beta_i}\hat{+}\delta_{w_i})$$

When $n = k+1$, there is

$$\mu_{\alpha_{k+1}}\bullet\mu_{w_{k+1}} \leq \mu_{\beta_{k+1}}\bullet\mu_{w_{k+1}},$$

$$\delta_{\alpha_{k+1}}\hat{+}\delta_{w_{k+1}} \geq \delta_{\beta_{k+1}}\hat{+}\delta_{w_{k+1}};$$

So, we know from $n = 2$

$$(\hat{+}_{i=1}^{n}(\mu_{\alpha_i}\bullet\mu_{w_i}))\hat{+}(\mu_{\alpha_{k+1}}\bullet\mu_{w_{k+1}}) \leq (\hat{+}_{i=1}^{n}(\mu_{\beta_i}\bullet\mu_{w_i}))\hat{+}(\mu_{\beta_{k+1}}\bullet\mu_{w_{k+1}}),$$

$$(\bullet_{i=1}^{n}(\delta_{\alpha_i}\hat{+}\delta_{w_i}))\bullet(\delta_{\alpha_{k+1}}\hat{+}\delta_{w_{k+1}}) \geq (\bullet_{i=1}^{n}(\delta_{\beta_i}\hat{+}\delta_{w_i}))\bullet(\delta_{\beta_{k+1}}\hat{+}\delta_{w_{k+1}});$$

Because

$$DFRMA_w(\alpha) = (\hat{+}_{i=1}^{k+1}(\mu_{\alpha_i}\bullet\mu_{w_i}), \bullet_{i=1}^{k+1}(\delta_{\alpha_i}\hat{+}\delta_{w_i})),$$

$$DFRMA_w(\beta) = (\hat{+}_{i=1}^{k+1}(\mu_{\beta_i}\bullet\mu_{w_i}), \bullet_{i=1}^{k+1}(\delta_{\beta_i}\hat{+}\delta_{w_i}));$$

We can see from Definition 4 that $DFRMA_w(\alpha) \leq DFRMA_w(\beta)$. □

Proposition 3 (Boundary). *Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a set of disturbing fuzzy numbers, where $\alpha_i = (\mu_{\alpha_i}, \delta_{\alpha_i})$, $i = 1, 2, \cdots, n$; The disturbing fuzzy number $w_i = (\mu_{w_i}, \delta_{w_i})$ is the weight of α_i, and $w = (w_1, w_2, \cdots, w_n)$ is the weight vector, let*

$$x_1 = \hat{+}_{i=1}^{n}(\min(\mu_{\alpha_i})\bullet\mu_{w_i}), y_2 = \bullet_{i=1}^{n}(\max(\delta_{\alpha_i})\bullet\mu_{w_i}),$$

$$x_2 = \hat{+}_{i=1}^{n}(\max(\mu_{\alpha_i})\bullet\mu_{w_i}), y_1 = \bullet_{i=1}^{n}(\min(\delta_{\alpha_i})\bullet\mu_{w_i}),$$

and $\alpha^- = (x_1, y_2)$, $\alpha^+ = (x_x, y_1)$, then

$$\alpha^- \leq DFRMA_w(\alpha) \leq \alpha^+.$$

Proof. $\forall i$, $i = 1, 2, \cdots, n$, there is

$$\min(\mu_{\alpha_i}) \leq \mu_{\alpha_i} \leq \max(\mu_{\alpha_i}),$$

$$\min(\delta_{\alpha_i}) \leq \delta_{\alpha_i} \leq \max(\delta_{\alpha_i}).$$

So

$$\hat{+}_{i=1}^{n}(\mu_{\alpha_i}\bullet\mu_{w_i}) \geq \hat{+}_{i=1}^{n}(\min(\mu_{\alpha_i})\bullet\mu_{w_i}) = x_1,$$

$$\bullet_{i=1}^{n}(\mu_{\alpha_i}\hat{+}\mu_{w_i}) \leq \bullet_{i=1}^{n}(\min(\mu_{\alpha_i})\hat{+}\mu_{w_i}) = y_2,$$

$$\hat{+}_{i=1}^{n}(\mu_{\alpha_i}\bullet\mu_{w_i}) \leq \hat{+}_{i=1}^{n}(\max(\mu_{\alpha_i})\bullet\mu_{w_i}) = x_2,$$

$$\bullet_{i=1}^{n}(\mu_{\alpha_i}\hat{+}\mu_{w_i}) \geq \bullet_{i=1}^{n}(\min(\mu_{\alpha_i})\hat{+}\mu_{w_i}) = y_1,$$

We can see from Definition 4 that $\alpha^- \leq DFRMA_w(\alpha) \leq \alpha^+$. □

4. Disturbing Fuzzy Multi-Attribute Decision-Making Method Based on DFRMA Operator

A complete decision is usually made up of five parts: decision-maker, alternative, attribute, attribute information, and weight information. Disturbing fuzzy multi-attribute decision-making means that the number of decision alternatives is limited, and the attribute information is represented by a disturbing fuzzy number. The disturbing fuzzy multi-attribute decision-making problem is given a limited set of alternatives and the data value of each option under the corresponding attribute through some scientific and reasonable methods or theories to comprehensively evaluate these alternatives and then select the most suitable solution or choose a solution that is most satisfactory to decision-makers.

In the fields of management, economy, and military, such as personnel promotion, investment decision-making, and military system effectiveness evaluation, sometimes decision-makers cannot give accurate attribute weight information or decision-maker weight information, or even the value range of the above two weight information. At this time, the use of disturbing fuzzy numbers to represent the weight information is particularly important. Based on the advantages of the DFRMA operator, we propose a disturbing fuzzy multi-attribute decision-making method based on the DFRMA operator, which is used to solve the problems in the above decision theory.

Let $Y = \{Y_1, \cdots, Y_n\}$: a discrete set of n feasible alternatives; $G = \{G_1, \cdots, G_m\}$: a finite set of attributes, whose weight vector is $w = (w_1, \cdots, w_m)$, where the weights are known and disturbing fuzzy number; $A = (a_{ij})_{n\times m}$ be the disturbing fuzzy decision matrix, where $a_{ij} = (\mu_{a_{ij}}, \delta_{a_{ij}})$ is the attribute value expressed by the disturbing fuzzy number, and $\mu_{a_{ij}}$ indicates the overall degree that the attribute G_j by the alternative Y_i, and $\delta_{a_{ij}}$ indicates decision-maker consider the degree of "disturbing" by the subjective or objective difference in the cognition of the attribute G_j by the alternative Y_i. Our goal is to obtain the comprehensive attribute value of the alternative and rank its size. Specific steps are as follows:

Step 1: In order to eliminate the impact of different types of attributes on the final decision result, the benefit-type attribute and cost-type attribute in the disturbing fuzzy decision matrix $A = (a_{ij})_{n\times m}$ are normalized by Equations (13) and (14), and then the disturbing fuzzy normalized decision matrix $R = (r_{ij})_{n\times m}$ is obtained.

If the attribute G_j is a disturbing benefit-type attribute, making

$$r^{ij} = \left(\frac{a_\mu^{ij}}{\sum\limits_{i=1}^{n} a_\mu^{ij} + \sum\limits_{i=1}^{n} a_\delta^{ij}}, \frac{a_\delta^{ij}}{\sum\limits_{i=1}^{n} a_\mu^{ij} + \sum\limits_{i=1}^{n} a_\delta^{ij}}\right) \tag{13}$$

If the attribute G_j is a disturbing cost-type attribute, making

$$r^{ij} = \left(\frac{1/a_\delta^{ij}}{\sum\limits_{i=1}^{n} 1/a_\mu^{ij} + \sum\limits_{i=1}^{n} 1/a_\delta^{ij}}, \frac{1/a_\mu^{ij}}{\sum\limits_{i=1}^{n} 1/a_\mu^{ij} + \sum\limits_{i=1}^{n} 1/a_\delta^{ij}}\right) \tag{14}$$

Step 2: Using attribute weight $w = (w_1, w_2, \cdots, w_m)$ and DFRMA operator to aggregate the data in the disturbing fuzzy normalized decision matrix $R = (r_{ij})_{n\times m}$, the decision-maker's comprehensive evaluation value D_i for alternative Y_i is obtained.

Step 3: The comprehensive evaluation value D_i of alternative Y_i is ranked by using the Boolean matrix ranking method of disturbing fuzzy numbers (the detailed steps will be presented in the example) and the decision result is obtained.

If there are multiple decision-makers involved in the decision, you can proceed as follows.

Step 1: The benefit-type attribute and cost-type attribute in the disturbing fuzzy decision matrix $A^s = (a_{ij}^s)_{n \times m}$ given by the S-th place expert are normalized and converted into the disturbing fuzzy normalized decision matrix $R^s = (r_{ij}^s)_{n \times m}$ of the S-th place expert by Equations (13) and (14);

Step 2: Using attribute weight $w = (w_1, w_2, \cdots, w_m)$ and DFRMA operator to aggregate the data in the disturbing fuzzy normalized decision matrix $R^s = (r_{ij}^s)_{n \times m}$ of the S-th place expert, the comprehensive evaluation value D_i^s of the S-th place expert to the alternative Y_i is obtained;

Step 3: The expert weight $\xi = (\xi_1, \xi_2, \cdots, \xi_l)$ and DFRMA operator are used to aggregate the comprehensive evaluation D_i^s of all experts on alternative Y_i, and the comprehensive evaluation D_i of all experts on alternative Y_i is obtained;

Step 4: The comprehensive evaluation value D_i of alternative Y_i is ranked by using the Boolean matrix ranking method of disturbing fuzzy numbers and the decision result is obtained.

5. Example

5.1. Example Analysis

A university evaluates four colleges under the school $Y_i (i = 1, 2, 3, 4)$, and now invites an expert to evaluate Y_i. The evaluation indicators are G_1 (teaching), G_2 (scientific research), G_3 (administration), and G_4 (service); attribute weight $w = ((0.30, 0.10), (0.25, 0.15), (0.60, 0.05), (0.15, 0.10))$, disturbing fuzzy decision matrix given by experts:

$$A = \begin{bmatrix} (0.40, 0.25) & (0.50, 0.30) & (0.35, 0.25) & (0.40, 0.10) \\ (0.60, 0.30) & (0.45, 0.15) & (0.55, 0.25) & (0.35, 0.20) \\ (0.70, 0.55) & (0.60, 0.25) & (0.65, 0.20) & (0.25, 0.15) \\ (0.55, 0.30) & (0.40, 0.20) & (0.45, 0.40) & (0.30, 0.25) \end{bmatrix}$$

Ranking the four colleges:

Step 1: Because the attributes are the benefit attribute, it is unnecessary to normalize the disturbing fuzzy decision matrix;

Step 2: Using attribute weight $w = ((0.30, 0.10), (0.25, 0.15), (0.60, 0.05), (0.15, 0.10))$ and DFRMA operator to aggregate the data in the disturbing fuzzy decision matrix A, the decision-makers comprehensive evaluation value D_i for the alternative Y_i is obtained:

$$\begin{aligned} DFRMA_w((0.40, 0.25), (0.50, 0.30), (0.35, 0.25), (0.40, 0.10)) = (0.428, 0.005) = D_1 \\ DFRMA_w((0.60, 0.30), (0.45, 0.15), (0.55, 0.25), (0.35, 0.20)) = (0.538, 0.006) = D_2 \\ DFRMA_w((0.70, 0.25), (0.60, 0.25), (0.65, 0.20), (0.25, 0.15)) = (0.606, 0.005) = D_3 \\ DFRMA_w((0.55, 0.30), (0.40, 0.20), (0.45, 0.40), (0.30, 0.25)) = (0.476, 0.012) = D_4 \end{aligned}$$

Step 3: The possibility matrix P is constructed from Equation (10):

$$P = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \\ 1 & 0 & 0 & 0.5 \end{bmatrix};$$

The Boolean matrix Q is constructed from Equation (11):

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix};$$

Obtaining the ranking vector $\rho = (1,3,4,2)^T$, so $Y_3 \succ Y_2 \succ Y_4 \succ Y_1$; therefore, College Y_3 wins.

In the case that the alternatives, attributes, attribute weight, and the disturbing fuzzy decision matrix given by the first expert remain unchanged, suppose that the disturbing fuzzy matrix given by two more experts is

$$A^2 = \begin{bmatrix} (0.60, 0.30) & (0.65, 0.15) & (0.50, 0.10) & (0.55, 0.35) \\ (0.65, 0.25) & (0.70, 0.35) & (0.65, 0.40) & (0.25, 0.15) \\ (0.70, 0.25) & (0.65, 0.20) & (0.75, 0.20) & (0.35, 0.35) \\ (0.80, 0.35) & (0.50, 0.20) & (0.45, 0.20) & (0.55, 0.30) \end{bmatrix}$$

$$A^3 = \begin{bmatrix} (0.55, 0.35) & (0.60, 0.35) & (0.55, 0.15) & (0.45, 0.15) \\ (0.55, 0.20) & (0.65, 0.20) & (0.70, 0.15) & (0.30, 0.25) \\ (0.75, 0.25) & (0.70, 0.15) & (0.65, 0.10) & (0.25, 0.20) \\ (0.70, 0.30) & (0.55, 0.20) & (0.55, 0.25) & (0.35, 0.10) \end{bmatrix}$$

where the expert weight $\xi = ((0.60, 0.15), (0.55, 0.20), (0.70, 0.25))$ is to rank the four colleges.

Step 1: Because the attributes are the benefit attribute, it is unnecessary to normalize the disturbing fuzzy decision matrix;

Step 2: Using attribute weight $w = ((0.30, 0.10), (0.25, 0.15), (0.60, 0.05), (0.15, 0.10))$ and DFRMA operator to aggregate the data in the disturbing fuzzy decision matrix A^s, the comprehensive evaluation value D_i^s of alternative Y_i by the s-th place decision-maker is obtained;

$$\begin{aligned}
DFRMA_w^1((0.40, 0.25), (0.50, 0.30), (0.35, 0.25), (0.40, 0.10)) = (0.428, 0.005) = D_1^1 \\
DFRMA_w^1((0.60, 0.30), (0.45, 0.15), (0.55, 0.25), (0.35, 0.20)) = (0.538, 0.006) = D_2^1 \\
DFRMA_w^1((0.70, 0.25), (0.60, 0.25), (0.65, 0.20), (0.25, 0.15)) = (0.606, 0.005) = D_3^1 \\
DFRMA_w^1((0.55, 0.30), (0.40, 0.20), (0.45, 0.40), (0.30, 0.25)) = (0.476, 0.012) = D_4^1 \\
DFRMA_w^2((0.60, 0.30), (0.65, 0.15), (0.50, 0.10), (0.55, 0.35)) = (0.559, 0.005) = D_1^2 \\
DFRMA_w^2((0.65, 0.25), (0.70, 0.35), (0.65, 0.40), (0.25, 0.15)) = (0.610, 0.010) = D_2^2 \\
DFRMA_w^2((0.70, 0.25), (0.65, 0.20), (0.75, 0.20), (0.35, 0.35)) = (0.655, 0.007) = D_3^2 \\
DFRMA_w^2((0.80, 0.35), (0.50, 0.20), (0.45, 0.20), (0.55, 0.30)) = (0.555, 0.008) = D_4^2 \\
DFRMA_w^3((0.55, 0.35), (0.60, 0.35), (0.55, 0.15), (0.45, 0.15)) = (0.557, 0.006) = D_1^3 \\
DFRMA_w^3((0.55, 0.20), (0.65, 0.20), (0.70, 0.15), (0.30, 0.25)) = (0.613, 0.004) = D_2^3 \\
DFRMA_w^3((0.75, 0.25), (0.70, 0.15), (0.65, 0.10), (0.25, 0.20)) = (0.625, 0.003) = D_3^3 \\
DFRMA_w^3((0.70, 0.30), (0.55, 0.20), (0.55, 0.25), (0.35, 0.10) = (0.576, 0.009) = D_4^3
\end{aligned}$$

Step 3: The expert weight $\xi = ((0.60, 0.30), (0.55, 0.20), (0.70, 0.40))$ and DFRMA operator are used to aggregate the comprehensive evaluation value D_i^s of the s-th place expert on alternative D_i^s, and the comprehensive evaluation value D_i of all experts on alternative Y_i is obtained;

$$D_1 = (0.686, 0.025),\ D_2 = (0.743, 0.026),\ D_3 = (0.771, 0.025),\ D_4 = (0.704, 0.026)$$

Step 4: The possibility matrix P is constructed from Equation (10):

$$P = \begin{bmatrix} 0.50 & 0 & 0 & 0.29 \\ 1 & 0.50 & 0.23 & 0.85 \\ 1 & 0.77 & 0.50 & 1 \\ 0.71 & 0.15 & 0 & 0.50 \end{bmatrix};$$

The Boolean matrix Q is constructed from Equation (11):

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix};$$

Obtaining the ranking vector $\rho = (1,3,4,2)^T$, so $Y_3 \succ Y_2 \succ Y_4 \succ Y_1$; therefore, College Y_3 wins.

5.2. Comparative Analysis

In order to demonstrate the effectiveness and superiority of the proposed methods, we adopt the method of comparative analysis. Therefore, we use the ideas of Lin [20] to solve the examples given in this paper and compare and analyze the results obtained in this paper. Lin's method uses attribute value information to construct the objective function, and the requirements for attribute weights should not only meet the normalization but also meet the weight changes of each attribute within a specific range. MATLAB is used to solve linear programming problems to obtain the optimal attribute weight vector. Finally, the comprehensive attribute value of the alternative is obtained through a specific method. By solving the linear programming problem, we obtain the most weighted vector $w = (0.30, 0.10, 0.55, 0.05)$, and the comprehensive attribute values of the alternative are $D_1 = (0.383, 0.248)$, $D_2 = (0.545, 0.253)$, $D_3 = (0.640, 0.308)$, and $D_4 = (0.468, 0.343)$, respectively. Using the Boolean matrix ranking method of disturbing fuzzy numbers, the ranking result is $Y_3 \succ Y_2 \succ Y_4 \succ Y_1$. This is the same as the result obtained by the method proposed in this paper, which further shows the effectiveness of the method proposed in this paper.

However, the advantages of the method proposed in this paper are as follows: it can directly aggregate the attribute value information and the weight information and can directly obtain the comprehensive attribute value of the scheme without solving it by computer software. It not only greatly reduces the calculation time and simplifies the decision-making process but also retains the originality of the weight information. In addition, this method allows the weight information to be represented by the disturbing fuzzy number, which solves this kind of problem well, so it has important guiding significance and application value, and the information represented by the disturbing fuzzy number can better reflect the uncertainty of human thinking.

6. Conclusions

In this paper, a new disturbing fuzzy aggregation operator is proposed to solve the disturbing fuzzy multi-attribute decision-making problem with the weight of disturbing fuzzy numbers. Some excellent properties of the operator are studied, and the concrete steps to solve the problem are given. This method is not only simple in the calculation but can also make full use of known information and retain the originality of information.

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