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Relation-Theoretic Nonlinear Almost Contractions with an Application to Boundary Value Problems

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Abstract: This article investigates certain fixed-point results enjoying nonlinear almost contraction conditions in the setup of relational metric space. Some examples are constructed in order to indicate the profitability of our results. As a practical use of our findings, we demonstrate the existence of a unique solution to a specific first-order boundary value problem.

Keywords: Λ-preserving sequence; almost contractions; boundary value problems

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1. Introduction

The classical BCP and its applications are widely recognized. In recent years, this crucial result has been generalized by many researchers using different approaches (e.g., [1,2]). One of the natural generalizations of this result is almost contraction, which was introduced by Berinde [3]. The almost contraction covers the usual (Banach) contraction, Kannan mapping [4], Chatterjea mapping [5], Zamfirescu contraction [6] and a certain class of quasi-contractions [7]. It is evident from this generalization that an almost contraction map does not necessarily possess a unique fixed point. Nonetheless, the convergence of the Picard iteration sequence can be used to calculate the fixed points of an almost contraction map. Alfuraidan et al. [8] presented a nonlinear formulation of almost contraction. For deeper investigation on almost contractions, we refer to [9–13].

In contrast, Alam and Imdad [14] presented an inevitable expansion of the BCP in a complete MS provisioned with an amorphous relation. In the past few years, multiple fixed-point results have been proven involving various contractivity conditions in relational MS, e.g., [15–22] and references therein. These outcomes comprised relation-preserving contractions that continue to be weaker than the ordinary contractions, which are indeed intended to verify the relation-preserving elements only.

The intent of this article is to investigate a fixed-point theorem employing nonlinear almost contraction in the setup of relational MS. The underlying relation in our results is amorphous (i.e., arbitrary), but the uniqueness theorem requires that the image of ambient space must be Λ^s-directed. This indicates the worth of our main results ahead of the results of Berinde [3], Alam and Imdad [14], Algehyne et al. [21], Khan [22] and Alfuraidan et al. [8]. We provide two illustrative examples that corroborate our results. In order to show the extent to the applicability of our results, we compute a unique solution of a first-order BVP.

2. Preliminaries

A relation Λ on a set $\mathbb{V}$ means any subset of $\mathbb{V}^2$. Assuming, $\mathbb{V}$ is a set, ζ is a metric on $\mathbb{V}$, Λ is a relation on $\mathbb{V}$ and $\mathcal{P} : \mathbb{V} \to \mathbb{V}$ is a function.

Definition 1 ([14]). *The elements $v, w \in \mathbb{V}$ are termed as Λ-comparative, denoted by $[v, w] \in \Lambda$, if $(v, w) \in \Lambda$ or $(w, v) \in \Lambda$.*

Definition 2 ([23]). *$\Lambda^{-1} := \{(v, w) \in \mathbb{V}^2 : (w, v) \in \Lambda\}$ is outlined as a transpose of Λ.*

Definition 3 ([23]). *The relation $\Lambda^s := \Lambda \cup \Lambda^{-1}$ is denoted as a symmetric closure of Λ.*

Proposition 1 ([14]). *$(v, w) \in \Lambda^s \Longleftrightarrow [v, w] \in \Lambda$.*

Definition 4 ([14]). *Λ is denoted as $\mathcal{P}$-closed if $(\mathcal{P}v, \mathcal{P}w) \in \Lambda$, whenever $(v, w) \in \Lambda$.*

Proposition 2 ([16]). *Λ is $\mathcal{P}^\imath$-closed provided Λ remains $\mathcal{P}$-closed.*

Definition 5 ([14]). *A sequence $\{v_\imath\} \subset \mathbb{V}$ verifying $(v_\imath, v_{\imath+1}) \in \Lambda$, $\forall\, \imath \in \mathbb{N}$ is denoted as Λ-preserving.*

Definition 6 ([15]). *$(\mathbb{V}, \zeta)$ is termed as Λ-complete whenever every Λ-preserving Cauchy sequence in $\mathbb{V}$ remains convergent.*

Definition 7 ([15]). *$\mathcal{P}$ is called Λ-continuous if for every $r \in \mathbb{V}$, $\mathcal{P}(v_\imath) \xrightarrow{\zeta} \mathcal{P}(r)$, whenever any Λ-preserving sequence $\{v_\imath\} \subset \mathbb{V}$ with $v_\imath \xrightarrow{\zeta} r$.*

Remark 1. *Completeness (respectively, continuity) implies Λ-completeness (respectively, Λ-continuity), but not the other way around.*

Definition 8 ([14]). *Λ is referred to as ζ-self-closed if every Λ-preserving convergent sequence in $\mathbb{V}$ permits a subsequence, every term of which remains Λ-comparative with the limit.*

Definition 9 ([24]). *A set $\mathbb{U} \subseteq \mathbb{V}$ is denoted as Λ-directed if for every $v, w \in \mathbb{U}$, $\exists\, u \in \mathbb{V}$ with $(v, u) \in \Lambda$ and $(w, u) \in \Lambda$.*

Following Bianchini and Grandolfi [25], we shall denote by Φ the family of the monotonically increasing functions $\varphi : [0, \infty) \to [0, \infty)$ with $\sum_{\imath=1}^{\infty} \varphi^\imath(t) < \infty, \quad \forall\, t \in (0, \infty)$.

Remark 2. *Each $\varphi \in \Phi$ verifies the following:*

(i) $\varphi(t) < t$, $\forall\, t \in (0, \infty)$;
(ii) $\lim_{r \to 0^+} \varphi(r) = \varphi(0) = 0$.

Inspired by Berinde [3], Alfuraidan et al. [8] introduced the class of functions $\theta : [0, \infty) \to [0, \infty)$ with $\lim_{r \to 0^+} \theta(r) = \theta(0) = 0$. In the following, we will denote this class by Θ.

Using the symmetry of metric ζ, one can put forth the following assertion.

Proposition 3. *If $\varphi \in \Phi$ and $\theta \in \Theta$, then the contractivity conditions listed below are identical:*

(i) $\zeta(\mathcal{P}v, \mathcal{P}w) \leq \varphi(\zeta(v, w)) + \theta(\zeta(w, \mathcal{P}v))$, $\forall\, (v, w) \in \Lambda$;
(ii) $\zeta(\mathcal{P}v, \mathcal{P}w) \leq \varphi(\zeta(v, w)) + \theta(\zeta(w, \mathcal{P}v))$, $\forall\, [v, w] \in \Lambda$.

3. Main Results

Herein, we present the fixed-point results under a new contractivity condition depending on the auxiliary functions belonging to classes Φ and Θ in the setup of relational MS.

Theorem 1. *Assume that $(\mathbb{V}, \zeta)$ is an MS endowed with a relation Λ and $\mathcal{P} : \mathbb{V} \to \mathbb{V}$ is a map. The following assumptions are also made:*

(a) $(\mathbb{V}, \zeta)$ *remains* Λ*-complete MS;*
(b) $\exists\, v_0 \in \mathbb{V}$ *verifying* $(v_0, \mathcal{P} v_0) \in \Lambda$*;*
(c) Λ *is* $\mathcal{P}$*-closed;*
(d) $\mathcal{P}$ *serves as* Λ*-continuous or* Λ *remains* ζ*-self-closed;*
(e) $\exists\, \varphi \in \Phi$ *and* $\theta \in \Theta$ *verifying*

$$\zeta(\mathcal{P} v, \mathcal{P} w) \leq \varphi(\zeta(v, w)) + \theta(\zeta(w, \mathcal{P} v)),\ \forall\ (v, w) \in \Lambda.$$

Then, $\mathcal{P}$ *admits a fixed point.*

Proof. Construct the sequence $\{\mathsf{v}_\imath\} \subset \mathbb{V}$ such that

$$\mathsf{v}_\imath = \mathcal{P}^\imath(\mathsf{v}_0) = \mathcal{P}(\mathsf{v}_{\imath-1}),\ \forall\ \imath \in \mathbb{N}. \tag{1}$$

Following assumption (b), the $\mathcal{P}$-closedness of Λ and Proposition 2, we find

$$(\mathcal{P}^\imath \mathsf{v}_0, \mathcal{P}^{\imath+1} \mathsf{v}_0) \in \Lambda$$

which, utilizing (1), becomes

$$(\mathsf{v}_\imath, \mathsf{v}_{\imath+1}) \in \Lambda, \quad \forall\ \imath \in \mathbb{N}. \tag{2}$$

Hence, $\{\mathsf{v}_\imath\}$ remains a Λ-preserving sequence.

Denote $\zeta_\imath := \zeta(\mathsf{v}_\imath, \mathsf{v}_{\imath+1})$. Applying the condition (e) to (2) and utilizing (1), we find

$$\zeta(\mathsf{v}_\imath, \mathsf{v}_{\imath+1}) \quad \leq \quad \varphi(\zeta(\mathsf{v}_{\imath-1}, \mathsf{v}_\imath)) + \theta(\zeta(\mathsf{v}_\imath, \mathcal{P}\mathsf{v}_{\imath-1})) = \varphi(\zeta(\mathsf{v}_{\imath-1}, \mathsf{v}_\imath)) + \theta(0),$$

i.e.,

$$\zeta_\imath \leq \varphi(\zeta_{\imath-1}),\ \forall\ \imath \in \mathbb{N},$$

which, by simple induction and the incensing property of φ, becomes

$$\zeta_\imath \leq \varphi^\imath(\zeta_0), \quad \forall\ \imath \in \mathbb{N}. \tag{3}$$

For every $\imath, \jmath \in \mathbb{N}$ with $\imath < \jmath$, using (3) and triangular inequality, we find

$$\begin{aligned}
\zeta(\mathsf{v}_\imath, \mathsf{v}_\jmath) &\leq \zeta_\imath + \zeta_{\imath+1} + \zeta_{\imath+2} + \cdots + \zeta_{\jmath-1} \\
&\leq \varphi^\imath(\zeta_0) + \varphi^{\imath+1}(\zeta_0) + \varphi^{\imath+2}(\zeta_0) + \cdots + \varphi^{\jmath-1}(\zeta_0) \\
&= \sum_{\kappa=\imath}^{\jmath-1} \varphi^\kappa(\zeta_0) \\
&\leq \sum_{\kappa \geq \imath} \varphi^\kappa(\zeta_0) \\
&\to 0 \text{ as } \imath \text{ (and hence } \jmath) \to \infty.
\end{aligned}$$

This verifies that $\{\mathsf{v}_\imath\}$ is Cauchy. As $\{\mathsf{v}_\imath\}$ also remains an Λ-preserving sequence, according to the Λ-completeness of $\mathbb{V}$, $\exists\, \mathsf{v}^* \in \mathbb{V}$ with $\mathsf{v}_\imath \xrightarrow{\zeta} \mathsf{v}^*$.

Now, we will conclude the proof by verifying that v^* remains a fixed point of $\mathcal{P}$. According to (d), first assume $\mathcal{P}$ is Λ-continuous. As $\{\mathsf{v}_\imath\}$ is a Λ-preserving sequence with $\mathsf{v}_\imath \xrightarrow{\zeta} \mathsf{v}^*$, we therefore have

$$\mathsf{v}_{\imath+1} = \mathcal{P}(\mathsf{v}_\imath) \xrightarrow{\zeta} \mathcal{P}(\mathsf{v}^*).$$

Making use of the uniqueness of the limit, we find $\mathcal{P}(\mathsf{v}^*) = \mathsf{v}^*$. In the alternative, we assume that Λ is ζ-self-closed. As $\{\mathsf{v}_\imath\}$ is a Λ-preserving sequence with $\mathsf{v}_\imath \xrightarrow{\zeta} \mathsf{v}^*$, $\exists$

a subsequence $\{\mathsf{v}_{\imath_k}\}$ of $\{\mathsf{v}_\imath\}$ verifying $[\mathsf{v}_{\imath_k}, \mathsf{v}^*] \in \Lambda, \quad \forall\, k \in \mathbb{N}$. Set $\varsigma_\imath := \zeta(\mathsf{v}^*, \mathsf{v}_\imath)$. Using assumption (e), Proposition 3 and $[\mathsf{v}_{\imath_k}, \mathsf{v}^*] \in \Lambda$, we find

$$\begin{aligned} \zeta(\mathsf{v}_{\imath_k+1}, \mathcal{P}\mathsf{v}^*) &= \zeta(\mathcal{P}\mathsf{v}_{\imath_k}, \mathcal{P}\mathsf{v}^*) \\ &\leq \varphi(\zeta(\mathsf{v}_{\imath_k}, \mathsf{v}^*)) + \theta(\zeta(\mathsf{v}^*, \mathcal{P}\mathsf{v}_{\imath_k})) \\ &= \varphi(\varsigma_{\imath_k}) + \theta(\varsigma_{\imath_k+1}). \end{aligned} \tag{4}$$

Now, $\mathsf{v}_{\imath_k} \xrightarrow{\zeta} \mathsf{v}^*$ implies that $\varsigma_{\imath_k} \longrightarrow 0^+$ in $[0, \infty)$, whenever $k \to \infty$. Therefore, upon letting $k \to \infty$ in (4) and employing the items (i) and (iii) of Remark 2 and the definition of Θ, we find

$$\begin{aligned} \lim_{k\to\infty} \zeta(\mathsf{v}_{\imath_k+1}, \mathcal{P}\mathsf{v}^*) &\leq \lim_{k\to\infty} \varphi(\varsigma_{\imath_k}) + \lim_{k\to\infty} \theta(\varsigma_{\imath_k+1}) \\ &= \lim_{t\to 0^+} \varphi(t) + \lim_{t\to 0^+} \theta(t) \\ &= 0 \end{aligned}$$

such that $\mathsf{v}_{\imath_k+1} \xrightarrow{\zeta} \mathcal{P}(\mathsf{v}^*)$, thereby implying $\mathcal{P}(\mathsf{v}^*) = \mathsf{v}^*$. Hence, in each of these cases, v^* serves as a fixed point of $\mathcal{P}$. □

Theorem 2. *Assume that all premises of Theorem 1 are valid. Furthermore, if*

(i) $\exists\, \varphi_0 \in \Phi$ *and* $\theta_0 \in \Theta$ *verifies*

$$\zeta(\mathcal{P}v, \mathcal{P}w) \leq \varphi_0(\zeta(v, w)) + \theta_0(\zeta(v, \mathcal{P}v)),\ \forall\, (v, w) \in \Lambda$$

and

(ii) $\mathcal{P}(\mathbb{V})$ *is* Λ^s*-directed,*

then $\mathcal{P}$ *possesses a unique fixed point.*

Proof. In lieu of Theorem 1, taking $\mathsf{v}, \mathsf{w} \in F(\mathcal{P})$, one obtains

$$\mathcal{P}^\imath(\mathsf{v}) = \mathsf{v} \text{ and } \mathcal{P}^\imath(\mathsf{w}) = \mathsf{w}, \quad \forall\, \imath \in \mathbb{N}. \tag{5}$$

As $\mathsf{v}, \mathsf{w} \in \mathcal{P}(\mathbb{V})$, according to hypothesis (ii), $\exists\, \mathsf{u} \in \mathbb{V}$ with $[\mathsf{v}, \mathsf{u}] \in \Lambda$ and $[\mathsf{w}, \mathsf{u}] \in \Lambda$, which, in view of the $\mathcal{P}$-closedness of Λ and Proposition 2, becomes

$$[\mathcal{P}^\imath\mathsf{v}, \mathcal{P}^\imath\mathsf{u}] \in \Lambda \quad \text{and} \quad [\mathcal{P}^\imath\mathsf{w}, \mathcal{P}^\imath\mathsf{u}] \in \Lambda, \quad \forall\, \imath \in \mathbb{N}. \tag{6}$$

Denote $\delta_\imath := \zeta(\mathcal{P}^\imath\mathsf{v}, \mathcal{P}^\imath\mathsf{u})$. We will prove that

$$\lim_{\imath\to\infty} \delta_\imath = \lim_{\imath\to\infty} \zeta(\mathcal{P}^\imath\mathsf{v}, \mathcal{P}^\imath\mathsf{u}) = 0. \tag{7}$$

Using (5), (6), assumption (ii) and the symmetry of ζ, we find

$$\begin{aligned} \zeta(\mathcal{P}^{\imath+1}\mathsf{v}, \mathcal{P}^{\imath+1}\mathsf{u}) &\leq \varphi_0(\zeta(\mathcal{P}^\imath\mathsf{v}, \mathcal{P}^\imath\mathsf{u})) + \theta_0(\zeta(\mathcal{P}^\imath\mathsf{v}, \mathcal{P}^{\imath+1}\mathsf{v})) \\ &= \varphi_0(\zeta(\mathcal{P}^\imath\mathsf{v}, \mathcal{P}^\imath\mathsf{u})) + \theta_0(0) \end{aligned}$$

such that

$$\delta_{\imath+1} \leq \varphi_0(\delta_\imath). \tag{8}$$

If $\delta_{\imath_0} = 0$ for some $\imath_0 \in \mathbb{N}$, then we have $\mathcal{P}^{\imath_0}(\mathsf{v}) = \mathcal{P}^{\imath_0}(\mathsf{u})$, thereby implying $\mathcal{P}^{\imath_0+1}(\mathsf{v}) = \mathcal{P}^{\imath_0+1}(\mathsf{u})$. Consequently, we obtain $\delta_{\imath_0+1} = 0$. By simple induction on $\imath$, we conclude $\delta_\imath = 0,\ \forall\, \imath \geq \imath_0$, thereby implying $\lim\limits_{\imath\to\infty} \delta_\imath = 0$. If $\delta_\imath > 0,\ \forall\, \imath \in \mathbb{N}$, then by simple induction on $\imath$ and increasing the property of φ_0 in (8), we find

$$\delta_{\imath+1} \leq \varphi_0(\delta_\imath) \leq \varphi_0^2(\delta_{\imath-1}) \leq \cdots \leq \varphi_0^\imath(\delta_1)$$

such that

$$\delta_{\imath+1} \leq \varphi_0^{\imath}(\delta_1).$$

Letting $\imath \to \infty$ in the above and utilizing the property of φ_0, we have

$$\lim_{\imath\to\infty} \delta_{\imath+1} \leq \lim_{\imath\to\infty} \varphi_0^{\imath}(\delta_1) = 0.$$

Therefore, in each case, (7) is verified. Similarly, we can verify that

$$\lim_{\imath\to\infty} \zeta(\mathcal{P}^{\imath}\mathrm{w}, \mathcal{P}^{\imath}\mathrm{u}) = 0. \tag{9}$$

On utilizing (7), (9) and the triangular inequality, we conclude

$$\zeta(\mathrm{v}, \mathrm{w}) = \zeta(\mathcal{P}^{\imath}\mathrm{v}, \mathcal{P}^{\imath}\mathrm{w}) \leq \zeta(\mathcal{P}^{\imath}\mathrm{v}, \mathcal{P}^{\imath}\mathrm{u}) + \zeta(\mathcal{P}^{\imath}\mathrm{u}, \mathcal{P}^{\imath}\mathrm{w}) \to 0 \quad \text{as} \quad \imath \to \infty$$

so $\mathrm{v} = \mathrm{w}$. This concludes the proof. □

Remark 3. *Under trivial relational $\Lambda = \mathbb{V}^2$ in Theorems 1 and 2, we obtain the nonlinear formulation of the result of Berinde [3], which runs as follows:*

Corollary 1. *Assume that $(\mathbb{V}, \zeta)$ is a complete MS and $\mathcal{P} : \mathbb{V} \to \mathbb{V}$ is a map. If $\exists\ \varphi \in \Phi$ and $\theta \in \Theta$ enjoy*

$$\zeta(\mathcal{P}v, \mathcal{P}w) \leq \varphi(\zeta(v, w)) + \theta(\zeta(w, \mathcal{P}v)),\ \forall\ v, w \in \mathbb{V},$$

then $\mathcal{P}$ admits a fixed point. In addition, if $\exists\ \varphi_0 \in \Phi$ and $\theta_0 \in \Theta$ verifies

$$\zeta(\mathcal{P}v, \mathcal{P}w) \leq \varphi_0(\zeta(v, w)) + \theta_0(\zeta(v, \mathcal{P}v)),\ \forall\ v, w \in \mathbb{V},$$

then $\mathcal{P}$ possesses a unique fixed point.

Remark 4. *Taking $\theta(t) = 0$ in Theorems 1 and 2, we find the results of Algehyne et al. [21].*

Remark 5. *If we take $\varphi(t) = kt,\ 0 < k < 1$ and $\theta(t) = Lt,\ L \geq 0$, then we derive the results of Khan [22].*

Remark 6. *Under the restriction $\Lambda = V(G)$ and $\varphi(t) = kt,\ 0 < k < 1$, our results reduce to the results of Alfuraidan et al. [8].*

Remark 7. *On setting $\varphi(t) = kt,\ 0 < k < 1$ and $\theta(t) = 0$, Theorems 1 and 2 deduce the corresponding results of Alam and Imdad [14].*

4. Illustrative Examples

This section is devoted to furnishing some examples in support of Theorems 1 and 2.

Example 1. *Let $\mathbb{V} = [0, 1]$ with metric $\zeta(v, w) = |v - w|$ and relation $\Lambda :=\geq$. Then, $(\mathbb{V}, \zeta)$ is a Λ-complete MS. Define a map $\mathcal{P} : \mathbb{V} \to \mathbb{V}$ by*

$$\mathcal{P}(v) = \begin{cases} 0, & \text{if } v = 1 \\ 2/3, & \text{otherwise.} \end{cases}$$

Naturally, $\mathcal{P}$ is Λ-continuous and Λ is $\mathcal{P}$-closed.

Define the functions $\varphi(t) = 2t/3$ and $\theta(t) = 3t/4$. Then, $\varphi \in \Phi$ and $\theta \in \Theta$. For any $(v, w) \in \Lambda$, we conclude

$$\zeta(\mathcal{P}v, \mathcal{P}w) \leq \varphi(\zeta(v, w)) + \theta(\zeta(w, \mathcal{P}v));$$

i.e., $\mathcal{P}$ verifies premise (e) of Theorem 1. Therefore, all the hypotheses of Theorem 1 are satisfied. Similarly, we can verify all premises of Theorem 2; so $\mathcal{P}$ possesses a fixed point. Indeed, here, $\mathcal{P}$ admits a fixed point: $v^* = 2/3$.

Example 2. *Let $\mathbb{V} = [0,1]$ with metric $\zeta(v,w) = |v - w|$ and relation $\Lambda = \mathbb{R} \times \mathbb{Q}$. Then, $(\mathbb{V}, \zeta)$ serves as a Λ-complete MS. Let $\mathcal{P}$ be considered as an identity map on $\mathbb{V}$. Naturally, $\mathcal{P}$ is Λ-continuous and Λ is $\mathcal{P}$-closed.*

Fix $\delta \in [0,1)$ and define the functions $\varphi(t) = \delta t$ and $\theta(t) = t - \delta t$. Then, $\varphi \in \Phi$ and $\theta \in \Theta$. For all $v, w \in \mathbb{V}$ satisfying $(v,w) \in \Lambda$, we have

$$\zeta(\mathcal{P}v, \mathcal{P}w) \leq \varphi(\zeta(v,w)) + \theta(\zeta(w, \mathcal{P}v));$$

i.e., $\mathcal{P}$ verifies premise (e) of Theorem 1. Therefore, all premises of Theorem 1 are satisfied. Consequently, $\mathcal{P}$ possesses a fixed point. Moreover, Theorem 2 cannot be applied to this example. Indeed, here, the entire $[0,1]$ forms the fixed point set.

5. An Application to BVP

Consider the following first-order periodic BVP:

$$\begin{cases} v'(\ell) = \digamma(\ell, v(\ell)), & \ell \in [0,a] \\ v(0) = v(a) \end{cases} \tag{10}$$

where $\digamma \in \mathcal{C}([0,a] \times \mathbb{R})$.

Definition 10 ([26])**.** *One says that $\underline{v} \in \mathcal{C}^1[0,a]$ serves as a lower solution of* (10) *if*

$$\begin{cases} \underline{v}'(\ell) \leq \digamma(\underline{v}, \underline{v}(\ell)), & \ell \in [0,a] \\ \underline{v}(0) \leq \underline{v}(a). \end{cases}$$

Definition 11 ([26])**.** *One says that $\overline{v} \in \mathcal{C}^1[0,a]$ serves as an upper solution of* (10) *if*

$$\begin{cases} \overline{v}'(\ell) \geq \digamma(\overline{v}, \overline{v}(\ell)), & \ell \in [0,a] \\ \overline{v}(0) \geq \overline{v}(a). \end{cases}$$

In the following, we will prove a result which guarantees the existence of a unique solution to problem (10).

Theorem 3. *In contrast with problem* (10)*, if $\exists\, k > 0$ and $\varphi \in \Phi$ verifying $\forall\, \alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$,*

$$0 \leq \digamma(\ell, \beta) + k\beta - [\digamma(\ell, \alpha) + k\alpha] \leq k\varphi(\beta - \alpha). \tag{11}$$

Then, (10) *possesses a unique solution to problem* (10) *whenever $\exists$ a lower solution of* (10)*.*

Proof. Express the problem (10) as

$$\begin{cases} v'(\ell) + kv(\ell) = \digamma(\ell, v(\ell)) + kv(\ell), & \forall\, \ell \in [0,a] \\ v(0) = v(a). \end{cases} \tag{12}$$

Thus, (12) is identical to the integral equation

$$v(\ell) = \int_0^a \Omega(\ell, \tau)[\digamma(\tau, v(\tau)) + kv(\tau)]d\tau \tag{13}$$

where $\Omega(\ell,\tau)$ is a Green function defined by

$$\Omega(\ell,\tau)=\begin{cases}\frac{e^{k(a+\tau-\ell)}}{e^{ka}-1}, & 0\leq\tau<\ell\leq a\\ \frac{e^{k(\tau-\ell)}}{e^{ka}-1}, & 0\leq\ell<\tau\leq a.\end{cases}$$

Set $\mathbb{V}:=\mathcal{C}[0,a]$. Consider the mapping $\mathcal{P}:\mathbb{V}\to\mathbb{V}$ defined by

$$(\mathcal{P}v)(\ell)=\int_0^a\Omega(\ell,\tau)[F(\tau,v(\tau))+kv(\tau)]d\tau,\quad\forall\,\ell\in[0,a]. \tag{14}$$

Consequently, $v\in\mathbb{V}$ continues to be a fixed point of $\mathcal{P}$ iff $v\in\mathcal{C}^1[0,a]$ becomes a solution of (13) and thereby (10).

On $\mathbb{V}$, define a metric ζ and a relation Λ given as:

$$\zeta(v,\omega)=\sup_{\ell\in[0,a]}|v(\ell)-\omega(\ell)|,\quad\forall\,v,\omega\in\mathbb{V} \tag{15}$$

and

$$\Lambda=\{(v,\omega):v(\ell)\leq\omega(\ell),\ \forall\,\ell\in[0,a]\}. \tag{16}$$

Now, we will approve each of the hypotheses of Theorem 1:

(i) Obviously, $(\mathbb{V},\zeta)$ is a Λ-complete MS.

(ii) Let $\ell\in\mathcal{C}^1[0,a]$ be a lower solution of (10); then, one has

$$\underline{v}'(\ell)+k\underline{v}(\ell)\leq F(\ell,\underline{v}(\ell))+k\underline{v}(\ell)\ \ \forall\,\ell\in[0,a].$$

Multiplying with $e^{k\ell}$, one obtains

$$(\underline{v}(\ell)e^{k\ell})'\leq[F(\ell,\underline{v}(\ell))+k\underline{v}(\ell)]e^{k\ell}\ \ \forall\,\ell\in[0,a]$$

thereby yielding

$$\underline{v}(\ell)e^{k\ell}\leq\underline{v}(0)+\int_0^\ell[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]e^{k\tau}d\tau,\ \ \forall\,\ell\in[0,a]. \tag{17}$$

Owing to $\underline{v}(0)\leq\underline{v}(a)$, one has

$$\underline{v}(0)e^{ka}\leq\underline{v}(a)e^{ka}\leq\underline{v}(0)+\int_0^a[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]e^{k\tau}d\tau$$

such that

$$\underline{v}(0)\leq\int_0^a\frac{e^{k\tau}}{e^{ka}-1}[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]d\tau. \tag{18}$$

According to (17) and (18), one finds

$$\begin{aligned}\underline{v}(\ell)e^{k\ell}&\leq\int_0^a\frac{e^{k\tau}}{e^{ka}-1}[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]d\tau+\int_0^\ell e^{k\tau}[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]d\tau\\&=\int_0^\ell\frac{e^{k(a+\tau)}}{e^{ka}-1}[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]d\tau+\int_\ell^a\frac{e^{k\tau}}{e^{ka}-1}[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]d\tau\end{aligned}$$

such that

$$\begin{aligned}\underline{v}(\ell)&\leq\int_0^\ell\frac{e^{k(a+\tau-\ell)}}{e^{ka}-1}[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]d\tau+\int_\ell^a\frac{e^{k(\tau-\ell)}}{e^{ka}-1}[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]d\tau\\&=\int_0^a\Omega(\ell,\tau)[F(\tau,\underline{v}(\tau))+k\underline{v}(\tau)]d\tau\\&=(\mathcal{P}\underline{v})(\ell),\ \forall\,\ell\in[0,a]\end{aligned}$$

which yields $(\underline{v}, \mathcal{P}\underline{v}) \in \Lambda$.

(iii) Take $(v, \omega) \in \Lambda$. Using (11), one has

$$F(\ell, v(\ell)) + kv(\ell) \leq F(\ell, \omega(\ell)) + k\omega(\ell), \quad \forall\, \ell \in [0, a]. \tag{19}$$

According to (14) and (19) and due to $\Omega(\ell, \tau) > 0$, $\forall\, (\ell, \tau) \in [0, a] \times [0, a]$, one obtains

$$\begin{aligned}
(\mathcal{P}v)(\ell) &= \int_0^a \Omega(\ell, \tau)[F(\tau, v(\tau)) + kv(\tau)]d\tau \\
&\leq \int_0^a \Omega(\ell, \tau)[F(\tau, \omega(\tau)) + k\omega(\tau)]d\tau \\
&= (\mathcal{P}\omega)(\ell) \quad \forall\, \ell \in [0, a],
\end{aligned}$$

which, by using (16), yields $(\mathcal{P}v, \mathcal{P}\omega) \in \Lambda$. Hence, Λ is $\mathcal{P}$-closed.

(iv) Suppose that $\{v_\imath\} \subset \mathbb{V}$ is a Λ-preserving sequence and it converges to $v \in \mathbb{V}$. Then, $\{v_\imath(\ell)\}$ for each $\ell \in [0, a]$ is a monotonically increasing sequence that converges to $v(\ell)$. This concludes that $v_\imath(\ell) \leq v(\ell)\ \forall\, \imath \in \mathbb{N}_0$ and $\ell \in [0, a]$. Again, according to (16), it follows that $(v_\imath, v) \in \Lambda$, $\forall\, \imath \in \mathbb{N}$, and hence Λ remains ζ-self-closed.

(v) Let $v, \omega \in \mathbb{V}$ such that $(v, \omega) \in \Lambda$. Using (11), (14) and (15), we find

$$\begin{aligned}
\zeta(\mathcal{P}v, \mathcal{P}\omega) &= \sup_{\ell \in [0,a]} |(\mathcal{P}v)(\ell) - (\mathcal{P}\omega)(\ell)| = \sup_{\ell \in [0,a]} \big((\mathcal{P}\omega)(\ell) - (\mathcal{P}v)(\ell)\big) \\
&\leq \sup_{\ell \in [0,a]} \int_0^a \Omega(\ell, \tau)[F(\tau, \omega(\tau)) + k\omega(\tau) - F(\tau, v(\tau)) - kv(\tau)]d\tau \\
&\leq \sup_{t \in I} \int_0^a \Omega(\ell, \tau)k\varphi(\omega(\tau) - v(\tau))d\tau.
\end{aligned} \tag{20}$$

Now, $0 \leq \omega(\tau) - v(\tau) \leq \zeta(v, \omega))$. Using the monotonicity of φ, one obtains $\varphi(\omega(\tau) - v(\tau)) \leq \varphi(\zeta(v, \omega))$, and hence (20) becomes

$$\begin{aligned}
d(\mathcal{P}v, \mathcal{P}\omega) &\leq k\varphi(d(v, \omega)) \sup_{\ell \in [0,a]} \int_0^a \Omega(\ell, \tau)d\tau \\
&= k\varphi(d(v, \omega)) \sup_{\ell \in [0,a]} \frac{1}{e^{ka} - 1}\left(\frac{1}{k}e^{k(a+\tau-\ell)}\Big]_0^t + \frac{1}{k}e^{k(\tau-\ell)}\Big]_\ell^a\right) \\
&= k\varphi(\zeta(v, \omega))\frac{1}{k(e^{ka} - 1)}(e^{ka} - 1) \\
&= \varphi(\zeta(v, \omega)).
\end{aligned}$$

It follows from $\forall\, v, \omega \in \mathbb{V}$ such that $(v, \omega) \in \Lambda$ that

$$\zeta(\mathcal{P}v, \mathcal{P}\omega) \leq \varphi(\zeta(v, \omega)) + \theta(\zeta(\omega, \mathcal{P}v))$$

and

$$\zeta(\mathcal{P}v, \mathcal{P}\omega) \leq \varphi(\zeta(v, \omega)) + \theta_0(\zeta(v, \mathcal{P}v)),$$

where $\theta \in \Theta$ and $\theta_0 \in \Theta$ are arbitrary. Thus, the contractivity conditions (e) (of Theorem 1) and (i) (of Theorem 2) hold. Let $v, \omega \in \mathbb{V}$ be arbitrary. Set $\mu := \max\{\mathcal{P}v, \mathcal{P}\omega\} \in \mathbb{V}$. As $(\mathcal{P}v, \mu) \in \Lambda$ and $(\mathcal{P}\omega, \mu) \in \Lambda$, $\{\mathcal{P}v, \varphi, \mathcal{P}\omega\}$ remains a path in Λ^s between $\mathcal{P}(v)$ and $\mathcal{P}(\omega)$. Thus, $\mathcal{P}(\mathbb{V})$ is Λ^s-directed. Thus, by Theorem 2, $\mathcal{P}$ possesses a unique fixed point, which serves as a unique solution to problem (10). $\square$

6. Conclusions

This manuscript comprised some fixed-point theorems under nonlinear almost contraction on an MS endowed with an amorphous relation. In the process, we also derived a nonlinear formulation of the Berinde fixed-point theorem [3]. Still, by utilizing our results, we can obtain several existing fixed-point theorems, especially thanks to Alam and Imdad [14], Algehyne et al. [21], Khan [22] and Alfuraidan et al. [8]. In future works, our results can be extended to nonlinear almost contractions by taking φ as a comparison function in the sense of Matkowski [27]. This work concludes the feasible application of the results proven herewith to a BVP, provided a lower solution exists. In a similar manner, readers can find an analogous result in the existence of an upper solution.

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Abbreviations

The following abbreviations and notations will be used in this manuscript:

MS	Metric space
BCP	Banach contraction principle
BVP	Boundary value problem
iff	If and only if
$\mathbb{N}$	The set of natural numbers
$\mathcal{C}(A)$	The class of all real-valued continuous functions on a set A
$\mathcal{C}^1(A)$	The class of all real-valued continuously differentiable functions on a set A

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