

Article

Statistical Solitonic Impact on Submanifolds of Kenmotsu Statistical Manifolds

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Abstract: In this article, we delve into the study of statistical solitons on submanifolds of Kenmotsu statistical manifolds, introducing the presence of concircular vector fields. This investigation is further extended to study the behavior of almost quasi-Yamabe solitons on submanifolds with both concircular and concurrent vector fields. Concluding our research, we offer a compelling example featuring a 5-dimensional Kenmotsu statistical manifold that accommodates both a statistical soliton and an almost quasi-Yamabe soliton. This example serves to reinforce and validate the principles discussed throughout our study.

Keywords: statistical soliton; almost quasi-Yamabe soliton; Kenmotsu statistical manifolds; Einstein statistical manifolds

MSC: 53C15; 53C25; 53C40

1. Introduction

Information geometry stands as a progressive and an interdisciplinary method in the realm of probability theory and statistical discussions. The information geometry, or affine geometry, and the hyperbolic geometry of the statistical manifolds are closely related. Actually, a Riemannian manifold $(\overline{\mathbb{B}}, g)$ is a statistical manifold of probability space in which the points represent probability distributions.

To obtain a geometric comprehension of statistical inference, let Ξ be a fixed-event space; let

$$\sigma(\Xi) = \left\{ \mathcal{P} : \Xi \longrightarrow \mathbb{R} : \int_{\Xi} \mathcal{P}(x) \geq 0, \forall x \in \Xi \right\}$$

be its probability distribution, while $\Omega \subset \mathbb{R}^n$ is a parameter space on the n -dimensional smooth family on Ξ . Then, $(\overline{\mathbb{B}}, g)$ can be considered as a statistical manifold, where $\overline{\mathbb{B}}$ and the Riemannian metric g are defined as follows [1]:

$$\overline{\mathbb{B}} = \{ \mathcal{P}(x, \Theta) \in \sigma(\Xi) : \Theta = (\Theta_1, \dots, \Theta_n) \in \Omega \},$$

$$g = \sum \left\{ \int_{\Xi} \left(\frac{\partial \log \mathcal{P}(x, \Theta)}{\partial \Theta_i} \right) \left(\frac{\partial \log \mathcal{P}(x, \Theta)}{\partial \Theta_j} \right) \mathcal{P}(x, \Theta) \right\} d\Theta_i d\Theta_j.$$

Numerous studies have also addressed certain applications of statistical manifolds in information geometry. For example, the authors of [2,3] have presented an extension of the ergodic, mixing, and Bernoulli levels of the ergodic hierarchy for statistical models on curved manifolds, using elements of the information geometry. They have also presented an analytical computation of the asymptotic temporal behavior of the information geometric complexity (IGC) of finite dimensional Gaussian statistical manifolds in the presence of microcorrelations (correlations between microvariables).



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The concept of statistical manifolds was initially introduced by Amari in [1] and subsequently applied by Lauritzen in [4]. A statistical manifold, denoted as $(\overline{\mathbb{B}}, \overline{\nabla}, g)$, is defined as a Riemannian manifold $(\overline{\mathbb{B}}, g)$ endowed with a Riemannian metric g if this metric is symmetric and a pair of torsion-free affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ on $\overline{\mathbb{B}}$, satisfies the following condition:

$$g(\mathcal{E}, \mathcal{F}) = g(\overline{\nabla}_g \mathcal{E}, \mathcal{F}) + g(\mathcal{E}, \overline{\nabla}_g^* \mathcal{F}),$$

for any $\mathcal{E}, \mathcal{F}, \mathcal{G} \in \Gamma(T\overline{\mathbb{B}})$, where $\overline{\nabla}^*$ is referred to as the dual connection on $\overline{\mathbb{B}}$.

Remark 1. A few noteworthy observations about this statistical structure are as follows:

1. $\overline{\nabla} = (\overline{\nabla}^*)^*$.
2. $2\overline{\nabla}^g = \overline{\nabla} + \overline{\nabla}^*$,
where $\overline{\nabla}^g$ representing the Levi-Civita connection of g on $\overline{\mathbb{B}}$.
3. if $(\overline{\nabla}, g)$ forms a statistical structure on $\overline{\mathbb{B}}$, then $(\overline{\nabla}^*, g)$ also constitutes a statistical structure.

In a statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}, g)$, let $\overline{\mathcal{T}} \in \Gamma(T\overline{\mathbb{B}}^{(1,2)})$ be defined as $\overline{\mathcal{T}} = \overline{\nabla}^g - \overline{\nabla}^* = \frac{1}{2}(\overline{\nabla} - \overline{\nabla}^*)$. This gives rise to the following relationships:

$$\overline{\mathcal{T}}_{\mathcal{E}} \mathcal{F} = \overline{\mathcal{T}}_{\mathcal{F}} \mathcal{E} \quad \text{and} \quad g(\overline{\mathcal{T}}_{\mathcal{E}} \mathcal{F}, \mathcal{G}) = g(\mathcal{F}, \overline{\mathcal{T}}_{\mathcal{E}} \mathcal{G}),$$

for any $\mathcal{E}, \mathcal{F}, \mathcal{G} \in \Gamma(T\overline{\mathbb{B}})$. Conversely, if $\overline{\mathcal{T}}$ satisfies the above conditions, then the triple $(\overline{\mathbb{B}}, \overline{\nabla} = \overline{\nabla}^g + \overline{\mathcal{T}}, g)$ takes on the role of a statistical manifold, and we denote $\overline{\mathcal{T}}_{\mathcal{E}} \mathcal{F}$ as $\overline{\mathcal{T}}(\mathcal{E}, \mathcal{F})$.

The statistical curvature tensor field $\overline{\mathbb{S}}^{\overline{\nabla}, \overline{\nabla}^*} = \overline{\mathbb{S}}$ with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ in $(\overline{\mathbb{B}}, \overline{\nabla}, g)$, can be expressed as per [5]:

$$\overline{\mathbb{S}}(\mathcal{E}, \mathcal{F})\mathcal{G} = \overline{Rie}^g(\mathcal{E}, \mathcal{F})\mathcal{G} + [\overline{\mathcal{T}}_{\mathcal{E}}, \overline{\mathcal{T}}_{\mathcal{F}}]\mathcal{G} \tag{1}$$

valid for any $\mathcal{E}, \mathcal{F}, \mathcal{G} \in \Gamma(T\overline{\mathbb{B}})$. The symbol \overline{Rie}^g denotes the curvature tensor field with respect to $\overline{\nabla}^g$.

The differential geometry of the Kenmotsu manifold constitutes a valuable component of contact geometry, offering significant applications in various fields, including theoretical physics. This significance extends to its statistical counterpart—the Kenmotsu statistical manifold—which is of comparable importance to the original Kenmotsu manifold.

In Tanno’s classification of connected almost-contact metric manifolds with maximal-dimension automorphism groups, Kenmotsu [6] explored the third class: $\overline{B} \times_s \overline{M}$, where \overline{B} is a line and \overline{M} is a Kaehlerian manifold. Kenmotsu characterized these manifolds, and later they were recognized as Kenmotsu manifolds. Furuhata et al. [5] extended this by introducing Kenmotsu statistical manifolds, which were derived by imposing an affine connection on a Kenmotsu manifold. They outlined a method for constructing Kenmotsu statistical manifolds as warped products of a holomorphic statistical manifold [7] and a line. Many researchers have devoted their precious time to studying the statistical version of named differentiable manifolds, as described in [8].

Ricci solitons, Yamabe solitons, η -Ricci solitons, and almost quasi-Yamabe solitons represent natural extensions of Einstein metrics. Hamilton’s introduction of the Ricci flow and Yamabe flow in 1982 gained substantial prominence, with the Ricci flow being described by the partial differential equation [9] used to smooth out metric singularities. Ricci flow has become a powerful tool for studying Riemannian manifolds with positive or negative curvature. A Ricci soliton on a Riemannian manifold $(\overline{\mathbb{B}}, g)$ is a tuple $(g, \mathcal{E}, \lambda)$ that satisfies the following equation:

$$\overline{Ric} + \frac{1}{2}\mathcal{L}_{\mathcal{E}}g + \lambda g = 0, \tag{2}$$

where \overline{Ric} represents the Ricci tensor, $\mathcal{L}_{\mathcal{E}}$ is the Lie derivative along the direction of the vector field \mathcal{E} , and λ is a real scalar. Such a soliton can be categorized as shrinking, steady, or expanding if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively.

An extension of Ricci solitons in a manifold conceding with an arbitrary linear connection $\overline{\nabla}$, distinguished from the Levi-Civita connection of g , is explained in [10].

The statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}, g)$ is called Ricci-symmetric if the Ricci operator \overline{Q} with respect to $\overline{\nabla}$ (equivalently, the dual operator \overline{Q}^* with respect to $\overline{\nabla}^*$) is symmetric (for more details, see [10]).

Definition 1. A pair (\mathcal{E}, λ) is called a statistical soliton on a Ricci-symmetric statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}, g)$ if the triplet $(\mathcal{E}, \lambda, g)$ is $\overline{\nabla}$ -Ricci and \overline{R}^* -Ricci solitons, we have [10]

$$\overline{\nabla}\mathcal{E} + \overline{Q} + \lambda I = 0, \quad \text{and} \quad \overline{\nabla}^*\mathcal{E} + \overline{Q}^* + \lambda I = 0, \tag{3}$$

where $g(\overline{Q}\mathcal{E}, \mathcal{F}) = \overline{Ric}(\mathcal{E}, \mathcal{F})$ and $g(\overline{Q}^*\mathcal{E}, \mathcal{F}) = \overline{Ric}^*(\mathcal{E}, \mathcal{F})$, for all vector fields \mathcal{E}, \mathcal{F} on $\overline{\mathbb{B}}$, and \overline{Ric} and \overline{Ric}^* indicate the Ricci tensor fields with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively.

In the field of differential geometry, the Yamabe problem centers on the quest for Riemannian metrics characterized by a constant scalar curvature. This problem is named after the mathematician Hidehiko Yamabe, who first put forth this inquiry in 1960. Within the field of differential geometry, the Yamabe flow stands as an intrinsic geometric process that induces the deformation of the metric of a Riemannian manifold. Notably, the fixed points of the Yamabe flow correspond to metrics exhibiting a constant scalar curvature.

The notion of Yamabe solitons plays a pivotal role, giving rise to self-similar solutions in the context of the Yamabe flow, as highlighted in [9]. A Yamabe soliton is essentially a self-similar solution within the framework of the Yamabe flow.

When the dimension of the manifold is $n = 2$, the Yamabe flow coincides with the Ricci flow defined by Equation (2). However, for dimensions exceeding $n > 2$, the Yamabe flow and the Ricci flow do not align. This discrepancy arises from the fact that the Yamabe flow preserves the conformal class of the metric, whereas the Ricci flow does not hold this property in general.

A Riemannian manifold $(\overline{\mathbb{B}}, g)$ is known as a Yamabe soliton if it possesses a vector field \mathcal{E} satisfying:

$$\mathcal{L}_{\mathcal{E}}g = 2(\overline{R} - \lambda)g, \tag{4}$$

where λ is a real number. Moreover, the concept of Yamabe solitons corresponds to self-similar solutions of the Yamabe flow.

In their recent work published in [11], Chen and Deshmukh delved into the concept of quasi-Yamabe solitons. In the context of our present study, we expand upon this concept, encompassing a more general scenario in which the constants are treated as functions. If λ is a smooth function defined on the manifold $\overline{\mathbb{B}}$, then the metric satisfying Equation (4) is referred to as an almost Yamabe soliton [12].

Consider an n -dimensional Riemannian manifold $(\overline{\mathbb{B}}, g)$ with $n > 2$, where \mathcal{E} represents a vector field and η is a 1-form on $\overline{\mathbb{B}}$. We have

Definition 2. Let $(\overline{\mathbb{B}}, g)$ be an n -dimensional Riemannian manifold ($n > 2$), while \mathcal{E} represents a vector field and η represents a 1-form on $\overline{\mathbb{B}}$. An almost quasi-Yamabe soliton on $\overline{\mathbb{B}}$ is defined by the set $(g, \mathcal{E}, \lambda, \omega)$, which satisfies the equation [11]:

$$\frac{1}{2}\mathcal{L}_{\mathcal{E}}g + (\lambda - \overline{R})g + \omega\eta \otimes \eta = 0, \tag{5}$$

where λ and ω are smooth functions defined on $\overline{\mathbb{B}}$.

The theory of concircular vector fields on a Riemannian manifold $(\overline{\mathbb{B}}, g)$ was introduced by Fialkow in 1939 [13]. These vector fields adhere to the following condition:

$$\overline{\nabla}_{\mathcal{E}}v = \delta\mathcal{E}, \tag{6}$$

where $\mathcal{E} \in \Gamma(T\overline{\mathbb{B}})$ and $\overline{\nabla}$ represents the Levi-Civita connection. Notably, $T\overline{\mathbb{B}}$ denotes the tangent bundle of $\overline{\mathbb{B}}$, and δ stands for a non-trivial function on $\overline{\mathbb{B}}$. These concircular vector fields are sometimes referred to as geodesic fields due to the fact that their integral curves follow geodesic paths [13]. Additionally, Chen [14] conducted a study involving Ricci solitons on submanifolds of Riemannian manifolds equipped with concircular vector fields. In the specific instance when $\delta = 1$ in Equation (6), the concircular vector field v is known as a concurrent vector field.

The research of Ricci solitons, Yamabe solitons, and their variants in diverse geometric contexts has gained significant traction over the last two decades, with applications in fields such as general relativity, applied mathematics, and theoretical physics. These investigations have been extended to almost contact manifolds, including work by Nagaraja and Premalatha [15], Blaga [16,17], Calin [18], Danish [19,20], Aliya et al. [21–23], and others [24–26].

Given this backdrop, our study is motivated by a desire to extend Ricci solitons and Yamabe solitons to Kenmotsu statistical manifolds. We embark on establishing this novel framework by introducing these solitons in the context of a statistical constant curvature in the Kenmotsu statistical manifold.

2. Preliminaries

Let (N, ∇, g) be a statistical submanifold in $(\overline{\mathbb{B}}, \overline{\nabla}, g)$. Then, the Gauss formulae are given by [27]:

$$\overline{\nabla}_{\mathcal{E}}\mathcal{F} = \nabla_{\mathcal{E}}\mathcal{F} + h(\mathcal{E}, \mathcal{F}), \tag{7}$$

and

$$\overline{\nabla}_{\mathcal{E}}^*\mathcal{F} = \nabla_{\mathcal{E}}^*\mathcal{F} + h^*(\mathcal{E}, \mathcal{F}), \tag{8}$$

for any $\mathcal{E}, \mathcal{F} \in \Gamma(TN)$. We denote the dual connections on $\Gamma(TN^\perp)$ by D^\perp and $D^{\perp*}$. Then, the corresponding Weingarten formulae are as follows[27]:

$$\overline{\nabla}_{\mathcal{E}}U = -A_U\mathcal{E} + D_{\mathcal{E}}^\perp U, \tag{9}$$

and

$$\overline{\nabla}_{\mathcal{E}}^*U = -A_U^*\mathcal{E} + D_{\mathcal{E}}^{\perp*}U, \tag{10}$$

for any $\mathcal{E} \in \Gamma(TN)$ and $U \in \Gamma(TN^\perp)$. The embedding curvature tensors of N in $\overline{\mathbb{B}}$, which are symmetric and bilinear in nature, are represented as h and h^* respectively. The linear transformations A_U and A_U^* are precisely defined in [27] as

$$g(h(\mathcal{E}, \mathcal{F}), U) = g(A_U^*\mathcal{E}, \mathcal{F}), \tag{11}$$

and

$$g(h^*(\mathcal{E}, \mathcal{F}), U) = g(A_U\mathcal{E}, \mathcal{F}). \tag{12}$$

A submanifold (N, ∇, g) of a statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}, g)$ is totally umbilical if

$$h(\mathcal{E}, \mathcal{F}) = g(\mathcal{E}, \mathcal{F})H \tag{13}$$

and

$$h^*(\mathcal{E}, \mathcal{F}) = g(\mathcal{E}, \mathcal{F})H^* \tag{14}$$

for any vector fields $\mathcal{E}, \mathcal{F} \in \Gamma(TN)$. Moreover, if $h = 0$ and $h^* = 0$, then N is totally geodesic. Additionally, when $H = 0$ and $H^* = 0$, N is minimal in $\overline{\mathbb{B}}$. Also, N is referred to as U -umbilical with respect to a normal vector field U if $A_U = f I$ and $A_U^* = f I$, where f is a function on N and I stands for the identity map.

The Riemannian curvature tensor fields with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ are denoted by \overline{Rie} and \overline{Rie}^* , respectively. Furthermore, Rie and Rie^* symbolize the Riemannian curvature tensor fields in connection with the induced connections ∇ and ∇^* from $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively. As outlined in [27], the Gauss equations take the following form:

$$g(\overline{Rie}(\mathcal{E}, \mathcal{F})\mathcal{G}, \mathcal{H}) = g(Rie(\mathcal{E}, \mathcal{F})\mathcal{G}, \mathcal{H}) + g(h(\mathcal{E}, \mathcal{G}), h^*(\mathcal{F}, \mathcal{H})) - g(h^*(\mathcal{E}, \mathcal{H}), h(\mathcal{F}, \mathcal{G})), \tag{15}$$

and

$$g(\overline{Rie}^*(\mathcal{E}, \mathcal{F})\mathcal{G}, \mathcal{H}) = g(Rie^*(\mathcal{E}, \mathcal{F})\mathcal{G}, \mathcal{H}) + g(h^*(\mathcal{E}, \mathcal{G}), h(\mathcal{F}, \mathcal{H})) - g(h(\mathcal{E}, \mathcal{H}), h^*(\mathcal{F}, \mathcal{G})), \tag{16}$$

for any $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H} \in \Gamma(TN)$. Also, we have

$$2\overline{S} = \overline{Rie} + \overline{Rie}^*, \tag{17}$$

and

$$2S = Rie + Rie^*, \tag{18}$$

where $S^{\nabla, \nabla^*} = S \in \Gamma(TN^{(1,3)})$ denotes the statistical curvature tensor field with respect to ∇ and ∇^* of N .

In most cases, it is not possible to define sectional curvature using the standard definitions with respect to dual connections that might not satisfy the metric properties. Nevertheless, Opozda introduced a novel approach to defining sectional curvature on a statistical manifold, as described in [28,29]:

$$\begin{aligned} \overline{Sec}(\mathcal{E} \wedge \mathcal{F}) &= g(\overline{S}(\mathcal{E}, \mathcal{F})\mathcal{F}, \mathcal{E}) \\ &= \frac{1}{2}(g(\overline{Rie}(\mathcal{E}, \mathcal{F})\mathcal{F}, \mathcal{E}) + g(\overline{Rie}^*(\mathcal{E}, \mathcal{F})\mathcal{F}, \mathcal{E})), \end{aligned} \tag{19}$$

for any orthonormal vectors $\mathcal{E}, \mathcal{F} \in \Gamma(T\overline{\mathbb{B}})$.

Kenmotsu geometry constitutes a distinctive field in differential geometry, finding valuable applications in various domains such as the mechanics of dynamical systems with time-dependent Hamiltonians, geometrical optics, thermodynamics, and geometric quantization. Additionally, the examination of submanifolds in the framework of Kenmotsu ambient spaces is an essential aspect of Kenmotsu geometry, and it has garnered substantial attention from numerous geometers.

Definition 3 ([5]). Let $(\overline{\mathbb{B}}, \overline{\nabla}, g, \phi, \xi)$ be a Kenmotsu manifold. A quadruplet $(\overline{\mathbb{B}}, \overline{\nabla} = \overline{\nabla}^g + \overline{\mathcal{T}}, g, \phi, \xi)$ is called a Kenmotsu statistical manifold if $(\overline{\nabla}, g)$ is a statistical structure on $\overline{\mathbb{B}}$ and the formula

$$\overline{\mathcal{T}}(\mathcal{E}, \phi\mathcal{F}) = -\phi\overline{\mathcal{T}}(\mathcal{E}, \mathcal{F}) \tag{20}$$

holds for any $\mathcal{E}, \mathcal{F} \in \Gamma(T\overline{\mathbb{B}})$. Here, we describe $(\overline{\nabla}, g, \phi, \xi)$ as a Kenmotsu statistical structure on $\overline{\mathbb{B}}$.

Any $\mathcal{E} \in \Gamma(TN)$ can be decomposed uniquely into its tangent and normal parts $P\mathcal{E}$ and $C\mathcal{E}$, respectively,

$$\phi\mathcal{E} = P\mathcal{E} + C\mathcal{E}.$$

A statistical submanifold (N, ∇, g) in a Kenmotsu statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}, g, \phi, \xi)$ is called invariant when $C = 0$, or, in the case of being anti-invariant, when $P = 0$. In the former case, it signifies that $\phi\mathcal{E} \in \Gamma(TN)$ for any $\mathcal{E} \in \Gamma(TN)$; conversely, in the latter case, it implies that $\phi\mathcal{E} \in \Gamma(TN^\perp)$ for any $\mathcal{E} \in \Gamma(TN)$.

3. Essential Results for Kenmotsu Statistical Manifolds

Within this section, we revisit certain fundamental findings from [5], which will be essential for establishing the results presented in this article.

Proposition 1 ([5]). *Let (\tilde{B}, G, J) be an almost Hermitian manifold. Set $\overline{\mathbb{B}} = \tilde{B} \times \mathbb{R}$, $g = \exp^{2\alpha} g + (d\alpha)^2$, $\xi = \frac{\partial}{\partial \alpha} \in \Gamma(T\overline{\mathbb{B}})$ and define $\phi \in \Gamma(T\overline{\mathbb{B}}^{(1,1)})$ by $\phi\mathcal{E}_2 = J\mathcal{E}_2$ for any $\mathcal{E}_2 \in \Gamma(T\tilde{B})$ and $\phi\xi = 0$. Then,*

1. *The triple (g, ϕ, ξ) is an almost contact metric structure on $\overline{\mathbb{B}}$.*
2. *The pair (G, J) is a Kähler structure on \tilde{B} if and only if the triple (g, ϕ, ξ) is a Kenmotsu structure on $\overline{\mathbb{B}}$.*

Theorem 1 ([5]). *Let $(\overline{\mathbb{B}}, \overline{\nabla}, g)$ be a statistical manifold and (g, ϕ, ξ) an almost-contact metric structure on $\overline{\mathbb{B}}$. $(\overline{\nabla}, g, \phi, \xi)$ is a Kenmotsu statistical structure $\overline{\mathbb{B}}$ if and only if the following conditions hold:*

$$\overline{\nabla}_{\mathcal{E}}(\phi\mathcal{F}) - \phi\overline{\nabla}_{\mathcal{E}}^*\mathcal{F} = -\eta(\mathcal{F})\phi\mathcal{E} + g(\phi\mathcal{E}, \mathcal{F})\xi, \tag{21}$$

$$\overline{\nabla}_{\mathcal{E}}\xi = \mathcal{E} - [\eta(\mathcal{E}) - \mu(\mathcal{E})]\xi, \tag{22}$$

for any $\mathcal{E}, \mathcal{F} \in \Gamma(T\overline{\mathbb{B}})$, where $\mu(\mathcal{E}) = -\eta(\overline{\nabla}_{\mathcal{E}}^*\xi) = \eta(\overline{\nabla}_{\mathcal{E}}\xi) = \eta(\overline{\mathcal{T}}(\xi, \xi))\eta(\mathcal{E})$.

Proposition 2 ([5]). *Let $(\tilde{B}, \tilde{\nabla} = \nabla^G + \mathcal{T}, G, J)$ be a holomorphic statistical manifold, and $(\overline{\mathbb{B}} = \tilde{B} \times \mathbb{R}, g, \phi, \xi)$ the Kenmotsu manifold (as in Proposition 1). For any $\beta \in C^\infty(\overline{\mathbb{B}})$, define $\overline{\mathcal{T}} \in \Gamma(T\overline{\mathbb{B}}^{(1,2)})$ by*

$$\overline{\mathcal{T}}(\mathcal{E}_2, \mathcal{F}_2) = \mathcal{T}(\mathcal{E}_2, \mathcal{F}_2), \quad \overline{\mathcal{T}}(\mathcal{E}_2, \xi) = \overline{\mathcal{T}}(\xi, \mathcal{E}_2) = 0, \quad \text{and} \quad \overline{\mathcal{T}}(\xi, \xi) = \beta\xi.$$

Then, $(\overline{\nabla} = \nabla^g + \overline{\mathcal{T}}, g, \phi, \xi)$ is a Kenmotsu statistical structure on $\overline{\mathbb{B}}$.

4. Statistical Solitons on Submanifolds of Kenmotsu Statistical Manifolds

Consider the pair (ξ, λ) on (N, ∇, g) and let $\dim(N) = s$. This pair is labeled as a statistical soliton if the triple (g, ξ, λ) satisfies both ∇ -Ricci and ∇^* -Ricci soliton conditions, as defined in Equation (3). Consequently, referring to Equation (3), we obtain

$$g(\nabla_{\mathcal{E}}\xi, \mathcal{F}) + Ric^\nabla(\mathcal{E}, \mathcal{F}) + \lambda g(\mathcal{E}, \mathcal{F}) = 0, \tag{23}$$

where Ric^∇ signifies the Ricci curvature tensor of N with respect to ∇ .

Using Equation (7) and Theorem 1, we get

$$\mathcal{E} - [\eta(\mathcal{E}) - \mu(\mathcal{E})]\xi = \overline{\nabla}_{\mathcal{E}}\xi = \nabla_{\mathcal{E}}\xi + h(\mathcal{E}, \xi). \tag{24}$$

It is important to mention here that $\mu(\mathcal{E}) = \eta(\overline{\mathcal{T}})\eta(\mathcal{E}) = \beta\eta(\mathcal{E})$. Equation (24) becomes

$$\nabla_{\mathcal{E}}\xi + h(\mathcal{E}, \xi) = \mathcal{E} + (\beta - 1)\eta(\mathcal{E})\xi. \tag{25}$$

If ζ is tangent to N then, equating tangential and normal components of (25), we get

$$\nabla_{\mathcal{E}}\zeta = \mathcal{E} + (\beta - 1)\eta(\mathcal{E})\zeta \quad \text{and} \quad h(\mathcal{E}, \zeta) = 0. \tag{26}$$

The torsion tensor field of ∇ vanishes, that is $\nabla_{\mathcal{E}}\mathcal{F} - \nabla_{\mathcal{F}}\mathcal{E} = [\mathcal{E}, \mathcal{F}]$ and upon considering (23) and (26), we can deduce the following relation:

$$Ric^{\nabla}(\mathcal{E}, \mathcal{F}) = -(\lambda + 1)g(\mathcal{E}, \mathcal{F}) + (1 - \beta)\eta(\mathcal{E})\eta(\mathcal{F}), \tag{27}$$

which indicates that N is an η -Einstein submanifold.

Consequently, we can establish the subsequent result:

Theorem 2. *If the data (g, ζ, λ) show statistical soliton on a submanifold (N, ∇, g) of a Kenmotsu statistical manifold $(\mathbb{B}, \overline{\nabla}, g, \zeta)$ (as in Proposition 2) and ζ is tangent to N , then N is the η -Einstein manifold.*

Remark 2. *By examining the dual counterparts of the Equations (26) in Theorem 2, we achieve the following results: $\nabla_{\mathcal{E}}^*\zeta = \mathcal{E} + (\beta - 1)\eta(\mathcal{E})\zeta$ and $h^*(\mathcal{E}, \zeta) = 0$, which can be succinctly expressed as $H^* = 0$.*

Thus, we can also formulate the dual case, as follows:

Theorem 3. *If the data (g, ζ, λ) are a statistical soliton on a submanifold (N, ∇^*, g) of a Kenmotsu statistical manifold $(\mathbb{B}, \overline{\nabla}^*, g, \zeta)$ (as in Proposition 2) and ζ is tangent to N , then N is η -Einstein manifold.*

Now, employing the formula:

$$Rie^{\nabla}(\mathcal{E}, \mathcal{F})\zeta = \nabla_{\mathcal{E}}\nabla_{\mathcal{F}}\zeta - \nabla_{\mathcal{F}}\nabla_{\mathcal{E}}\zeta - \nabla_{[\mathcal{E}, \mathcal{F}]} \zeta. \tag{28}$$

By utilizing Equation (26), we arrive at

$$\begin{aligned} Rie^{\nabla}(\mathcal{E}, \mathcal{F})\zeta &= (1 - \beta)[\eta(\mathcal{E})\mathcal{F} - \eta(\mathcal{F})\mathcal{E}] + (1 - \beta)\eta([\mathcal{E}, \mathcal{F}])\zeta \\ &\quad + \mathcal{F}[(1 - \beta)\eta(\mathcal{E})] - \mathcal{E}[(1 - \beta)\eta(\mathcal{F})], \end{aligned} \tag{29}$$

which implies:

$$Ric^{\nabla}(\mathcal{E}, \zeta) = [(1 - \beta)(1 - s) + \sum_{i=1}^s e_i(\beta)]\eta(\mathcal{E}) + \mathcal{E}(\beta) \quad \text{for all } \mathcal{E}. \tag{30}$$

By substituting $\mathcal{F} = \zeta$ into (27) and using (30), we obtain

$$\lambda = -[\beta + (1 - s)(1 - \beta) + \sum_{i=1}^s e_i(\beta) + \zeta(\beta)] < 0$$

always. This leads to the subsequent outcome:

Theorem 4. *Let N be a submanifold of a Kenmotsu statistical manifold $(\mathbb{B}, \overline{\nabla}, g, \zeta)$ (as in Proposition 2) while ζ is tangent to N . Then, statistical soliton (g, ζ, λ) is always shrinking.*

In the case where ζ is normal to N , considering any $\mathcal{E} \in \Gamma(TN)$ and utilizing Equation (24), the result is

$$\nabla_{\mathcal{E}}\zeta = \mathcal{E} \quad \text{and} \quad h(\mathcal{E}, \zeta) = (\beta - 1)\eta(\mathcal{E})\zeta. \tag{31}$$

As a consequence, we have

$$g(\nabla_{\mathcal{E}}\xi, \mathcal{F}) + Ric^{\nabla}(\mathcal{E}, \mathcal{F}) + \lambda g(\mathcal{E}, \mathcal{F}) = 0. \tag{32}$$

Using equations (23), (31), and (32), we arrive at

$$Ric^{\nabla}(\mathcal{E}, \mathcal{F}) = -(\lambda + 1)g(\mathcal{E}, \mathcal{F}). \tag{33}$$

This indicates that N possesses Einstein properties.

Therefore, we can present the following result:

Theorem 5. *If the data (g, ξ, λ) show statistical soliton on a submanifold N of a Kenmotsu statistical manifold $(\mathbb{B}, \bar{\nabla}, g, \xi)$ (as in Proposition 2) and ξ is normal to N , then N is Einstein.*

By considering both of the Equations (31) in Theorem 5, we can also present the dual case in the following manner:

Theorem 6. *If the data (g, ξ, λ) show statistical soliton on a submanifold N of a Kenmotsu statistical manifold $(\mathbb{B}, \bar{\nabla}^*, g, \xi)$ (as in Proposition 2) and ξ is normal to N , then N is Einstein.*

Furthermore, from Equation (31), it follows that $Rie^{\nabla}(\mathcal{E}, \mathcal{F})\xi = 0$, which consequently yields $Ric^{\nabla}(\mathcal{E}, \mathcal{F}) = 0$. By utilizing (33), we can derive

$$Ric^{\nabla}(\mathcal{E}, \xi) = -(\lambda + 1)\eta(\mathcal{E}). \tag{34}$$

As a result, we find that $\lambda = -1 < 0$. This leads to the subsequent outcome:

Theorem 7. *Let N be a submanifold of a Kenmotsu statistical manifold $(\mathbb{B}, \bar{\nabla}, g, \xi)$ (as in Proposition 2) and ξ be normal to N . Then, statistical soliton (g, ξ, λ) is always shrinking.*

Remark 3. *Theorems 4 and 7 hold true for the dual counterpart.*

5. Statistical Solitons Featuring a Concircular Vector Field

In this section, we delve into the investigation of statistical solitons on submanifolds of the Kenmotsu statistical manifold, as outlined in Proposition 2: $(\mathbb{B}, \bar{\nabla} = \nabla^g + \bar{T}, g, \phi, \xi)$, taking into consideration the existence of a concircular vector field v .

Now, the concircular vector field v with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ is given by

$$\bar{\nabla}_{\mathcal{E}}v = \delta\mathcal{E}, \quad \bar{\nabla}_{\mathcal{E}}^*v = \delta\mathcal{E}, \tag{35}$$

where $\delta : \bar{B} \rightarrow \mathbb{R}$ is a smooth function.

To begin with, we obtain the following outcomes:

Lemma 1. *Let N be a submanifold of a Kenmotsu statistical manifold $(\mathbb{B}, \bar{\nabla}, g, \xi)$ with a concircular vector field on v . Then, N is v^{nor} -umbilical if and only if v^{tan} is a concircular vector field on N with respect to $\bar{\nabla}$.*

Proof. Since v is a concircular vector field on \mathbb{B} , we have $\bar{\nabla}_{\mathcal{E}}v = \delta\mathcal{E}$. Using (7) and (9), we get

$$\delta\mathcal{E} = \nabla_{\mathcal{E}}v^{tan} + h(\mathcal{E}, v^{tan}) - A_{v^{nor}}\mathcal{E} + D_{\mathcal{E}}^{\perp}v^{nor}, \tag{36}$$

for any vector field \mathcal{E} tangent to N . By comparing the tangential component in (36), we have

$$\nabla_{\mathcal{E}}v^{tan} = A_{v^{nor}}\mathcal{E} + \delta\mathcal{E}, \tag{37}$$

which shows that v^{tan} is a concircular vector field on N , such that $\nabla_{\mathcal{E}} v^{tan} = (f + \delta)\mathcal{E}$, since N is v^{nor} -umbilical.

Conversely, if v^{tan} is a concircular vector field on submanifold N , then there is a non-trivial function σ on N , such that

$$\nabla_{\mathcal{E}} v^{tan} = \sigma \mathcal{E}. \tag{38}$$

By comparing Equations (37) and (38), we get

$$A_{v^{nor}} \mathcal{E} = (\sigma - \delta)\mathcal{E}. \tag{39}$$

This (39) shows that N is v^{nor} -umbilical. \square

Remark 4. Now, by examining the dual forms of Equations (36) and (37), we derive the subsequent equations:

$$\delta \mathcal{E} = \nabla_{\mathcal{E}}^* v^{tan} + h^*(\mathcal{E}, v^{tan}) - A_{v^{nor}}^* \mathcal{E} + D_{\mathcal{E}}^{\perp} v^{nor} \tag{40}$$

$$\nabla_{\mathcal{E}}^* v^{tan} = A_{v^{nor}}^* \mathcal{E} + \delta \mathcal{E}. \tag{41}$$

Hence, we can also establish the dual version of Lemma 1:

Lemma 2. Let N be a submanifold of a Kenmotsu statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}^*, g, \xi)$ with a concircular vector field on v . Then, N is v^{nor} -umbilical if and only if v^{tan} is a concircular vector field on N .

Consider that v acts as a concircular vector field on $(\overline{\mathbb{B}}, \overline{\nabla} = \nabla^g + \overline{\mathcal{T}}, g, \phi, \xi)$,

$$v = v^{tan} + v^{nor}. \tag{42}$$

Based on (3) and (37), it can be inferred that (N, v^{tan}, λ, g) represents a statistical soliton if and only if

$$Ric(\mathcal{E}, \mathcal{F}) + \delta g(\mathcal{E}, \mathcal{F}) + g(h(\mathcal{E}, \mathcal{F}), v^{nor}) + \lambda g(\mathcal{E}, \mathcal{F}) = 0. \tag{43}$$

Subsequently, by employing (43), we obtain the following results:

Theorem 8. A submanifold N admits statistical soliton (g, v^{tan}, λ) in a Kenmotsu statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}, g, \xi)$, then the Ricci tensor of N satisfies

$$Ric(\mathcal{E}, \mathcal{F}) = -(\lambda + \delta)g(\mathcal{E}, \mathcal{F}) - g(h(\mathcal{E}, \mathcal{F}), v^{nor}). \tag{44}$$

for any vector fields \mathcal{E}, \mathcal{F} tangent to N .

Also, we demonstrate the duality of Theorem 8:

Theorem 9. A submanifold N admits a statistical soliton (g, v^{tan}, λ) in a Kenmotsu statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}^*, g, \xi)$, then the Ricci tensor of N satisfies

$$Ric^*(\mathcal{E}, \mathcal{F}) = -(\lambda + \delta)g(\mathcal{E}, \mathcal{F}) - g(h(\mathcal{E}, \mathcal{F}), v^{nor}). \tag{45}$$

Assuming a statistical soliton $(g, v^{tan}, \lambda, \omega)$ on a submanifold N of a Kenmotsu statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}, g, \xi)$ to be totally umbilical, we can deduce from Lemma 1 that v^{tan}

corresponds to a concircular vector field, that is, $\bar{\nabla}_{\mathcal{E}}v^{tan} = \sigma\mathcal{E}$. By combining (43) and (45) with (1), we can derive the ensuing pair of equations:

$$Ric(\mathcal{E}, \mathcal{F}) = -(\lambda + \sigma)g(\mathcal{E}, \mathcal{F}) \tag{46}$$

and

$$Ric^*(\mathcal{E}, \mathcal{F}) = -(\lambda + \sigma)g(\mathcal{E}, \mathcal{F}). \tag{47}$$

Equations (46) and (47) yield the ensuing theorems:

Theorem 10. *Let $(g, v^{tan}, \lambda, \omega)$ represent totally umbilical statistical soliton on a submanifold N of a Kenmotsu statistical manifold (as shown in Proposition 2) $(\bar{\mathbb{B}}, \bar{\nabla}, g, \xi)$. Then, N is isometric to a sphere and its quasi-Einstein.*

Theorem 11. *Let $(g, v^{tan}, \lambda, \omega)$ be totally umbilical statistical soliton on a submanifold N of a Kenmotsu statistical manifold (as in Proposition 2) $(\bar{\mathbb{B}}, \bar{\nabla}^*, g, \xi)$. Then, N is isometric to a sphere and its quasi-Einstein.*

6. Almost Quasi-Yamabe Soliton on Submanifolds of Kenmotsu Statistical Manifold

In this section, our assumptions revolve around the structure $(\bar{\mathbb{B}}, \bar{\nabla}^*, g, \xi)$, representing a Kenmotsu statistical manifold in accordance with Proposition 2, while also considering the presence of a concircular vector field v . Concurrently, let N be a submanifold in $\bar{\mathbb{B}}$. Notably, we designate the tangential and normal components of v as v^{tan} and v^{nor} , respectively.

Continuing in the same vein, given that v qualifies as a concircular vector field and making use of Equations (7) and (9), we are able to come to the following conclusion:

$$\delta\mathcal{E} = \nabla_{\mathcal{E}}v^{tan} + h(\mathcal{E}, v^{tan}) - A_{v^{nor}}\mathcal{E} + D_{\mathcal{E}}^{\perp}v^{nor}, \tag{48}$$

for any \mathcal{E} tangent to N . By comparing the tangential and normal components, we arrive at

$$\nabla_{\mathcal{E}}v^{tan} = A_{v^{nor}}\mathcal{E} + \delta\mathcal{E}, \quad h(\mathcal{E}, v^{tan}) = D_{\mathcal{E}}^{\perp}v^{nor}. \tag{49}$$

From the definition of Lie-derivative and (49), we have

$$(\mathcal{L}_{v^{tan}}g)(\mathcal{E}, \mathcal{F}) = 2\delta g(\mathcal{E}, \mathcal{F}) + 2g(A_{v^{nor}}\mathcal{E}, \mathcal{F}). \tag{50}$$

On combining (5) and (50), we find that

$$(R - \lambda - \delta)g(\mathcal{E}, \mathcal{F}) = g(A_{v^{nor}}\mathcal{E}, \mathcal{F}) - \omega\eta(\mathcal{E})\eta(\mathcal{F}). \tag{51}$$

As a result, we are now in a position to enunciate the following:

Theorem 12. *The almost quasi-Yamabe soliton $(g, v^{tan}, \lambda, \omega)$ on a submanifold N of a Kenmotsu statistical manifold $(\bar{\mathbb{B}}, \bar{\nabla}, g, \xi)$ (as in Proposition 2) satisfies*

$$(R - \lambda - \delta)g(\mathcal{E}, \mathcal{F}) = g(A_{v^{nor}}\mathcal{E}, \mathcal{F}) - \omega\eta(\mathcal{E})\eta(\mathcal{F}). \tag{52}$$

In the context of the dual case, an analogous theorem emerges:

Theorem 13. *The almost quasi-Yamabe soliton $(g, v^{tan}, \lambda, \omega)$ on a submanifold N of a Kenmotsu statistical manifold $(\bar{\mathbb{B}}, \bar{\nabla}^*, g, \xi)$ (as in Proposition 2) satisfies*

$$(R^* - \lambda - \delta)g(\mathcal{E}, \mathcal{F}) = g(A_{v^{nor}}^*\mathcal{E}, \mathcal{F}) - \omega\eta(\mathcal{E})\eta(\mathcal{F}). \tag{53}$$

Substituting $\mathcal{E} = \mathcal{F} = \zeta$ into (53) and considering the fact that N is minimal, we can employ (13) and (14) to deduce that

$$R = \lambda - \omega + \delta. \tag{54}$$

In light of the above, we can succinctly state the following results:

Theorem 14. *If an almost quasi-Yamabe soliton $(g, v^{tan}, \lambda, \omega)$ on a submanifold N of a Kenmotsu statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}, g, \zeta)$ (as in Proposition 2) is minimal, then $R = \lambda - \omega + \delta$.*

Theorem 15. *If an almost quasi-Yamabe soliton $(g, v^{tan}, \lambda, \omega)$ on a submanifold N of a Kenmotsu statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}^*, g, \zeta)$ (as in Proposition 2) is minimal, then $R = \lambda - \omega + \delta$.*

Presently, we can derive the subsequent corollaries specifically for the case in which $\delta = 1$, considering the concurrent vector field scenario:

Corollary 1. *If an almost quasi-Yamabe soliton $(g, v^{tan}, \lambda, \omega)$ on a submanifold N of a Kenmotsu statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}, g, \zeta)$ (as in Proposition 2) with the concurrent vector field is minimal, then $R = \lambda - \omega + 1$.*

Corollary 2. *If an almost quasi-Yamabe soliton $(g, v^{tan}, \lambda, \omega)$ on a submanifold N of a Kenmotsu statistical manifold $(\overline{\mathbb{B}}, \overline{\nabla}^*, g, \zeta)$ (as described in Proposition 2) with the concurrent vector field is minimal, then $R^* = \lambda - \omega + 1$.*

7. Some Examples

Example 1. *We examine a 5-dimensional Kenmotsu manifold as presented in [30]:*

$$(\overline{\mathbb{B}} = \{(x, y, z, u, v) \in \mathbb{R}^5 | v > 0\}, g, \phi, \zeta),$$

where the standard coordinates in \mathbb{R}^5 are denoted as (x, y, z, u, v) . We designate the vector fields v_1, v_2, v_3, v_4, v_5 as follows:

$$\begin{aligned} v_1 &= \exp^{-v} \frac{\partial}{\partial x}, & v_2 &= \exp^{-v} \frac{\partial}{\partial y}, \\ v_3 &= \exp^{-v} \frac{\partial}{\partial z}, & v_4 &= \exp^{-v} \frac{\partial}{\partial u}, \\ v_5 &= \frac{\partial}{\partial v} = \zeta. \end{aligned}$$

The Riemannian metric g is defined as

$$g(v_i, v_j) = 0, \quad g(v_i, v_i) = 1$$

for all $i \neq j$, where $i, j = 1, \dots, 5$. A $(1, 1)$ tensor field ϕ is introduced with the following components:

$$\begin{aligned} \phi(v_1) &= v_3, & \phi(v_2) &= v_4, & \phi(v_3) &= -v_1, \\ \phi(v_4) &= -v_2, & \phi(v_5) &= \phi(\zeta) = 0. \end{aligned}$$

The Levi-Civita connection ∇^g of g is determined through Koszul's formula:

$$\begin{aligned} \nabla_{v_i}^g v_i &= -\zeta, & \nabla_{v_i}^g v_j &= 0, \\ \nabla_{v_i}^g \zeta &= v_i & \nabla_{\zeta}^g v_i &= 0, \\ \nabla_{\zeta}^g \zeta &= 0, \end{aligned}$$

for all $i \neq j, i, j = 1, \dots, 4$.

Now, for any $\mathcal{E}, \mathcal{F} \in \Gamma(T\mathbb{B})$ and $a \in \mathbb{R}$, we define the difference tensor field $\overline{\mathcal{T}}$ as:

$$\overline{\mathcal{T}}(\mathcal{E}, \mathcal{F}) = ag(\mathcal{E}, \xi)g(\mathcal{F}, \xi)\xi.$$

Subsequently, the dual torsion-free affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are introduced as

$$\begin{aligned} \overline{\nabla}_{v_i}v_i &= -\xi, & \overline{\nabla}_{v_i}v_j &= 0, \\ \overline{\nabla}_{v_i}\xi &= v_i, & \overline{\nabla}_{\xi}v_i &= 0, \\ \overline{\nabla}_{\xi}\xi &= a\xi, \end{aligned}$$

and

$$\begin{aligned} \overline{\nabla}_{v_i}^*v_i &= -\xi, & \overline{\nabla}_{v_i}^*v_j &= 0, \\ \overline{\nabla}_{v_i}^*\xi &= v_i, & \overline{\nabla}_{\xi}^*v_i &= 0, \\ \overline{\nabla}_{\xi}^*\xi &= -a\xi. \end{aligned}$$

It can be verified that

$$\mathcal{G}g(\mathcal{E}, \mathcal{F}) = g(\overline{\nabla}_{\mathcal{G}}\mathcal{E}, \mathcal{F}) + g(\mathcal{E}, \overline{\nabla}_{\mathcal{G}}^*\mathcal{F}),$$

and

$$\overline{\mathcal{T}}(\mathcal{E}, \phi\mathcal{F}) + \phi\overline{\mathcal{T}}(\mathcal{E}, \mathcal{F}) = 0.$$

Consequently, the manifold $(\mathbb{B} = (x, y, z, u, v) \in \mathbb{R}^5 | v > 0, \overline{\nabla}, g, \phi, \xi)$ is established as a 5-dimensional Kenmotsu statistical manifold.

Example 2. Consider the Kenmotsu structure (ϕ, ξ, η, g) on the unit hypersphere S^5 . By setting $\overline{\mathcal{T}}(\mathcal{E}, \mathcal{F}) = a\eta(\mathcal{E})\eta(\mathcal{F})$ and $\overline{\nabla} = \nabla^g + \overline{\mathcal{T}}$, it becomes evident that $(S^5, \phi, \xi, \eta, g)$ satisfies the conditions of a Kenmotsu statistical manifold (see Example 1). Consequently, when $i = 1, 2$, let (x^i, y^i, v) denote the local coordinates of S^5 . The following assignments can be made:

1. $\xi = \frac{\partial}{\partial v}$,
2. $\phi\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$,
3. $\phi\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}$,
4. $\phi\left(\frac{\partial}{\partial v}\right) = 0$.

Now, consider a 3-dimensional submanifold $N = (x^1, y^1, 0, 0, v)$ in S^5 . It is important to note that N is invariant and ξ is tangent to N .

Example 3. Consider the upper half space $(\mathbb{H}^2 = (x, y) \in \mathbb{R}^2 | y > 0, g = y^{-2}(dx^2 + dy^2))$ with constant curvature -1 . An affine connection ∇ on \mathbb{H}^2 is defined as stated in [31]:

$$\begin{aligned} \nabla_{\partial_x}\partial_x &= 2y^{-1}\partial_y, \\ \nabla_{\partial_y}\partial_y &= y^{-1}\partial_y, \\ \nabla_{\partial_x}\partial_y &= \nabla_{\partial_y}\partial_x = 0. \end{aligned}$$

This setup defines a statistical manifold $(\mathbb{H}^2, \nabla, g)$ with constant curvature 0, making it a Ricci-flat statistical manifold. As a result, it serves as a steady Ricci soliton with $\lambda = 0$.

Example 4. Let us consider a statistical manifold $(M = (x, y) \in \mathbb{R}^2, \nabla, g = dx^2 + dy^2)$ with constant curvature -1 . An affine connection ∇ on M is provided according to [32], as follows:

$$\begin{aligned} \nabla_{\partial x} \partial x &= -\partial y, \\ \nabla_{\partial y} \partial y &= 0, \\ \nabla_{\partial x} \partial y &= \nabla_{\partial y} \partial x = \partial x. \end{aligned}$$

In addition, the conjugate connection ∇^* on M is defined as follows:

$$\begin{aligned} \nabla_{\partial x}^* \partial x &= -\partial y, \\ \nabla_{\partial y}^* \partial y &= 0, \\ \nabla_{\partial x}^* \partial y &= \nabla_{\partial y}^* \partial x = -\partial x. \end{aligned}$$

Consequently, the scalar curvature of M equals -2 , classifying (M, ∇, g) as an Einstein statistical manifold with $\lambda = -1$. Therefore, it can be characterized as a shrinking Ricci soliton with $\lambda < 0$.

Example 5. Consider an orthonormal frame field v_1, v_2, v_3 on a statistical manifold $(M = (x, y, z) \in \mathbb{R}^3, \nabla, g = dx^2 + dy^2 + dz^2)$. An affine connection ∇ on M can be expressed according to [33] as follows:

$$\begin{aligned} \nabla_{v_1} v_1 &= bv_1, \\ \nabla_{v_2} v_2 &= \nabla_{v_3} v_3 = \frac{b}{2} v_1, \\ \nabla_{v_1} v_2 &= \nabla_{v_2} v_1 = \frac{b}{2} v_2, \\ \nabla_{v_1} v_3 &= \nabla_{v_3} v_1 = \frac{b}{2} v_3, \\ \nabla_{v_2} v_3 &= \nabla_{v_3} v_2 = 0, \end{aligned}$$

where b represents a constant. Consequently, (M, ∇, g) stands as a statistical manifold with constant curvature $\frac{b^2}{4}$. The scalar curvature of M equates to $\frac{3b^2}{2}$. This configuration classifies it as an Einstein statistical manifold with $\lambda = \frac{b^2}{2}$. Therefore, it can be identified as an expanding Ricci soliton with $\lambda > 0$.

8. Illustration of Statistical and Almost Quasi-Yamabe Solitons on Kenmotsu Statistical Manifolds

Example 6. Consider a 5-dimensional Kenmotsu statistical manifold denoted as

$$(\overline{\mathbb{B}} = (x, y, z, u, v) \in \mathbb{R}^5 | v > 0, \overline{\nabla}, g, \phi, \zeta),$$

as illustrated in Example 1. In this context, the non-vanishing components of the Riemannian curvature Rie , the Ricci curvature Ric , and the scalar curvature tensor R concerning both $\overline{\nabla}$ and $\overline{\nabla}^*$ can be explicitly expressed as follows:

$$\begin{aligned} Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_1, v_2)v_1 &= v_2, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_1, v_2)v_2 = -v_1, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_1, v_3)v_1 = v_3, \\ Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_1, v_3)v_3 &= -v_1, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_1, v_4)v_1 = v_4, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_1, v_5)v_1 = v_5, \\ Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_1, v_4)v_4 &= -v_1, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_1, v_5)v_5 = -v_1, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_2, v_3)v_2 = v_3, \\ Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_2, v_3)v_3 &= -v_2, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_2, v_4)v_2 = v_4, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_2, v_4)v_4 = -v_2, \\ Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_2, v_5)v_2 &= v_5, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_2, v_5)v_5 = ae_2, Ric^{\overline{\nabla}, \overline{\nabla}^*}(v_3, v_4)v_3 = v_4, \end{aligned}$$

$$\begin{aligned} \text{Rie}^{\bar{\nabla}, \bar{\nabla}^*}(v_3, v_4)v_4 &= -v_3, & \text{Rie}^{\bar{\nabla}, \bar{\nabla}^*}(v_3, v_5)v_3 &= a\zeta, & \text{Rie}^{\bar{\nabla}, \bar{\nabla}^*}(v_3, v_5)v_5 &= ae_3, \\ \text{Rie}^{\bar{\nabla}, \bar{\nabla}^*}(v_4, v_5)v_4 &= a\zeta, & \text{Rie}^{\bar{\nabla}, \bar{\nabla}^*}(v_4, v_5)v_5 &= ae_4, & \text{Rie}^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_4)v_4 &= -v_1. \end{aligned}$$

Hence, from (17) and (18), we derive

$$\begin{aligned} S^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_2)v_1 &= v_2, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_2)v_2 &= -v_1, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_3)v_1 &= v_3, \\ S^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_3)v_3 &= -v_1, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_4)v_1 &= v_4, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_5)v_1 &= v_5, \\ S^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_4)v_4 &= -v_1, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_5)v_5 &= -v_1, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_2, v_3)v_2 &= v_3, \\ S^{\bar{\nabla}, \bar{\nabla}^*}(v_2, v_3)v_3 &= -v_2, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_2, v_4)v_2 &= v_4, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_2, v_4)v_4 &= -v_2, \\ S^{\bar{\nabla}, \bar{\nabla}^*}(v_2, v_5)v_2 &= v_5, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_2, v_5)v_5 &= ae_2, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_3, v_4)v_3 &= v_4, \\ S^{\bar{\nabla}, \bar{\nabla}^*}(v_3, v_4)v_4 &= -v_3, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_3, v_5)v_3 &= a\zeta, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_3, v_5)v_5 &= ae_3, \\ S^{\bar{\nabla}, \bar{\nabla}^*}(v_4, v_5)v_4 &= a\zeta, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_4, v_5)v_5 &= ae_4, & S^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_4)v_4 &= -v_1. \end{aligned}$$

$$\begin{aligned} \text{Ric}^{\bar{\nabla}, \bar{\nabla}^*}(v_1, v_1) &= \text{Ric}^{\bar{\nabla}, \bar{\nabla}^*}(v_2, v_2) = \text{Ric}^{\bar{\nabla}, \bar{\nabla}^*}(v_3, v_3) = \text{Ric}^{\bar{\nabla}, \bar{\nabla}^*}(v_4, v_4) = -4, \\ \text{Ric}^{\bar{\nabla}, \bar{\nabla}^*}(v_5, v_5) &= 3a + 1. \end{aligned}$$

Thus,

$$R^{\bar{\nabla}, \bar{\nabla}^*} = 3a - 15.$$

Regarding statistical solitons, utilizing (3), for $\lambda = 4$, the given data (g, ζ, λ) yield an expanding statistical soliton on the 5-dimensional Kenmotsu statistical manifold.

In the context of almost quasi-Yamabe solitons, considering Equation (5), we can set $\lambda = 3a - 16$ and $\omega = 1 - \beta$. Consequently, for the configuration $(g, \zeta, \lambda, \omega)$, this leads to the establishment of both an expanding and a shrinking almost quasi-Yamabe soliton. Specifically, these solitons emerge for the conditions of $a > 5$ and $a < 5$, respectively. This scenario unfolds in the setting of a Kenmotsu statistical manifold $(\mathbb{B} = (x, y, z, u, v) \in \mathbb{R}^5 | v > 0, \bar{\nabla}, g, \phi, \zeta)$.

9. Conclusions

The present work is a specialization of Amari’s theory of information geometry and statistical Riemannian manifolds. The study of the Kenmotsu manifold is an important part of contact geometry in differential geometry, with important applications in theoretical physics, among other areas. Its statistical equivalent, the Kenmotsu statistical manifold (see [34]), is also significant and is as important as the original Kenmotsu manifold. The interest from theoretical physicists has extended towards equations involving Ricci solitons and Yamabe solitons, particularly in the context of Einstein manifolds, quasi-Einstein manifolds, and string theory. In the study of Ricci and almost quasi-Yamabe solitons within geometric analysis, a crucial inquiry revolves around identifying the criteria that lead these entities to simplify into trivial Ricci solitons and trivial Yamabe solitons, respectively. Our findings represent significant advancements toward answering this question.

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