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Asymptotic Behavior of Stochastic Reaction–Diffusion Equations

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Abstract: In this paper, we concentrate on the propagation dynamics of stochastic reaction–diffusion equations, including the existence of travelling wave solution and asymptotic wave speed. Based on the stochastic Feynman–Kac formula and comparison principle, the boundedness of the solution of stochastic reaction–diffusion equations can be obtained so that we can construct a sup-solution and a sub-solution to estimate the upper bound and the lower bound of wave speed.

Keywords: stochastic reaction–diffusion equations; Feynman–Kac formula; travelling wave; asymptotic wave speed

MSC: 60H15; 37A25

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1. Introduction

In the current paper, we focus on stochastic reaction–diffusion equations driven by Gaussian noise

$$\begin{cases} du = [\triangle u + uc_1(u,v)]dt + \epsilon u dW_t, \\ dv = [\triangle v + vc_2(u,v)]dt + \epsilon v dW_t, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), \end{cases} \tag{1}$$

where u_0 and v_0 as initial data are both Heaviside functions. In this paper, it always holds that

(H1) $0 < \frac{\partial c_1(u,v)}{\partial v} < a$, $0 < \frac{\partial c_2(u,v)}{\partial u} < a$;
(H2) $c_1(u,v)$ is decreasing for u, $c_1(u,0) \le c_1(0,0)$, and $c_1(u,0) < 0$ for $u > \frac{1}{a}$; $c_2(u,v)$ is decreasing for v, $c_2(0,v) \le c_2(0,0)$, and $c_2(0,v) < 0$ for $v > \frac{1}{a}$;
(H3) There exists $\alpha \ge \beta > a$ for any $\xi \ge 0$, $u \ge 0$ and $v \ge 0$; it holds that $\beta\xi \le c_1(u,v) - c_1(u+\xi,v) \le \alpha\xi$, $\beta\xi \le c_2(u,v) - c_2(u,v+\xi) \le \alpha\xi$;
(H4) $c_i(0,0) > 2\epsilon^2$, $i = 1,2$.

In general, (H1) and (H2) imply that System (1) is cooperative, so its corresponding dynamical system is monotonic and a comparison method can be used in this system. (H4) ensures the noise is moderate, otherwise the solution of Equation (1) tends to zero as $t \to \infty$.

Under conditions (H1)–(H4), System (1) poses an only positive stable equilibrium denoted by (p_1, p_2). If $c_1(u,v) = c_1(u,0)$, $c_2(u,v) = c_2(0,v)$, Zhao and Øksendal [1] investigated the pathwise property and ergodicity of a stochastic reaction–diffusion equation in a scalar scale under conditions (H2)–(H4),

$$\begin{cases} du(t,x) = [\frac{D}{2}u_{xx}(t,x) + c(u(t,x))u(t,x))dt + k(t)u(t,x)dW_t, \\ u(0,x) = u_0(x). \end{cases} \tag{2}$$

Inspired by Benth and Gjessing [2], they successfully obtained the stochastic Feynman–Kac formula by constructing exponential martingale and revealed that the existence of a travelling wave solution and its asymptotic wave speed depend on the strength of noise. In detail, if $\liminf_{t\to\infty} \int_0^t k^2(s)ds > c_0 = c(0)$ and the noise is strong, the solution to Equation (2) almost surely tends to zero. If $\int_0^t k^2(s)ds < \infty$ and the noise is weak, the travelling wave solution of Equation (2) converges to the travelling wave solution of deterministic reaction–diffusion equation. Moreover, if $k_\infty = \lim_{t\to\infty} \int_0^t k^2(s)ds$ exist and the noise is moderate, the wavefront marker is $x = \sqrt{D(2c_0 - 2k_\infty)t}$.

If $\epsilon = 0$, Freidlin [3] studied the asymptotic behavior of the Cauchy problem under conditions (H1)–(H3),

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{D_1}{2}\frac{\partial^2 u}{\partial x^2} + c_{11}(u,v)u(t,x) + c_{12}v(t,x), \\ \frac{\partial v}{\partial t} = \frac{D_2}{2}\frac{\partial^2 v}{\partial x^2} + c_{21}(u,v)v(t,x) + c_{22}u(t,x), \\ u(0,x) = g_1(x), v(0,x) = g_2(x). \end{cases} \tag{3}$$

Until now, many papers have been concerned with stochastic travelling wave solutions in the scalar equation. Zhao [4] studied the wave speed of a stochastic KPP equation driven by white noise. Huang and Wang [5–9] investigated the asymptotic behavior of a stochastic reaction–diffusion equation driven by various noises. Indeed, a way to deal with the coupling terms is the crux in the research of travelling wave solution of stochastic reaction–diffusion equations. In this paper, with monotonic random dynamical system theory and comparison principle, the boundedness of solution to Equation (1) is obtained and used to construct a sup-solution and a sub-solution under conditions (H1)–(H4). Hence, via the SCP (Support Compactness Propagation) property proposed by Shiga [10] and two sufficient conditions proposed by Tribe [11], the existence of a travelling wave solution can be obtained. Again, with the boundedness of solution, we can estimate the upper bound and the lower bound of wave speed by a sup-solution and a sub-solution, respectively.

Throughout this paper, we set Ω as the space of temper distributions, $\mathcal{F}$ as the σ-algebra on Ω, and $(\Omega, \mathcal{F}, \mathbb{P})$ as the white noise probability space. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. We denote by $\phi_\lambda(x) = exp(-\lambda|x|)$. Here are some notations:

- $\mathcal{D}_{[0,\infty)} = \{\phi : R \to [0,\infty), \phi$ is right continuous and decreasing, $\phi_{-\infty} = \lim_{x\to\infty} \phi(x)$ exists$\}$;
- $\mathcal{D}_{[0,1]} = \{\phi : R \to [0,1], \phi$ is right continuous and decreasing$\}$;
- $\mathcal{D} = \{\phi \in \mathcal{D}_{[0,1]} : \phi(-\infty) = 1,\ \phi(\infty) = 0\}$;
- $C^+ = f|f : R \to [0,\infty)$ and f is continuous;
- $||f||_\lambda = \sup_{x\in R}(|f(x)\phi_\lambda(x)|)$;
- $C^+_\lambda = f \in C^+|f$ is continuous, and $|f(x)\phi_\lambda(x)| \to 0$ as $x \to \pm\infty$;
- $C^+_{tem} = \bigcap_{\lambda>0} C^+_\lambda$;
- $C^+_{C[0,1]} = \{f|f : R \to [0,1]\}$ is the space of nonnegative functions with compact support;
- $\Phi = f : ||f||_\lambda < \infty$ for some $\lambda < 0$ is the space of functions with exponential decay;
- $R_0(t) = \sup\{x \in R : u(t,x) > 0\}$ is the wavefront marker.

Lemma 1 ([11])**.** *A set $K \subset C^+_\lambda$ is called relatively compact if and only if*

(a) *K is equicontinuous on a compact set;*

(b) $\lim_{R\to\infty} \sup_{f\in K} \sup_{|x|\geq R} |f(x)e^{-\lambda|x|}| = 0.$

Lemma 2 ([11])**.** *$K \subset C^+_{tem}$ is (relatively) compact if and only if it is (relatively) compact in C^+_λ for all $\lambda > 0$.*

Lemma 3 ([11] **(Kolmogorov tightness criterion)).** *For $C < \infty, \delta > 0, \mu < \lambda, \gamma > 0$, we define*

$$K(C,\delta,\gamma,\mu) = \{f : |f(x) - f(x')| \le C|x-x'|^{\gamma} e^{\mu|x|} \text{ for all } |x-x'| \le \delta\},$$

then, with the above conditions, we know that $K(C,\delta,\gamma,\mu) \cap \{f : \int_R f(x)\phi_1 dx \le a\}$ is compact in C^+_λ, where a is a constant.

(1) If $\{X_n(\cdot)\}$ are C_λ-valued processes, with $\{\int_R X_n \phi_1 dx\}$ tight and there are $C_0 < \infty, p > 0, \gamma > 1, \mu < \lambda$ such that for all $n \ge 1$, $|x-y| \le 1$,

$$\mathbb{E}(|X_n(x) - X_n(y)|^p) \le C_0|x-y|^{\gamma} e^{\mu p|x|},$$

then $\{X_n\}$ are tight.

(2) Similarly, if $\{X_n\}$ are $C([0,T], C^+_\lambda)$-valued processes, with $\{\int_R X_n(0)\phi_1 dx\}$ tight, and there are $C_0 < \infty, p > 0, \gamma > 2, \mu < \lambda$ such that for all $n \ge 1, |x-y| \le 1, |t-t'| \le 1, t, t' \in [0,T]$,

$$\mathbb{E}(|X_n(x,t) - X_n(y,t')|^p) \le C_0(|x-y|^{\gamma} + |t-t'|^{\gamma}) e^{\mu p|x|},$$

then $\{X_n\}$ are tight.

2. Asymptotic Behavior of a Travelling Wave Solution

At the beginning of our work, we offer the definition of stochastic travelling wave solution in law. $\mathcal{D}_{[0,\infty)}$ values are equipped with the $L^1_{loc}(R)$ metric, $\mathcal{D}_{[0,1]}$ and $\mathcal{D}$ are measurable subsets of $\mathcal{D}_{[0,\infty)}$, and the three spaces are all Polish spaces and are compact. We consider a stochastic reaction–diffusion equation with a Heaviside function as follows:

$$\begin{cases} du = [\triangle u + f(u)]dt + \epsilon u dW_t, \\ u(0,x) = u_0(x). \end{cases} \tag{4}$$

Definition 1. *A stochastic travelling wave solution is a solution to $u = (u(t) : t \ge 0)$ to Equation (4) with values in $\mathcal{D}$ and for which the centered process $(\tilde{u} = u(t, \cdot + R_0(t)) : t \ge 0)$ is a stationary process with respect to time, and the law of a stochastic travelling wave solution is the law of $\tilde{u}(0)$ on $\mathcal{D}$.*

We denote by $Y = (u,v)^T$ and $F(Y) = (F_1(Y), F_2(Y)) = (uc_1(u,v), vc_2(u,v))^T$; then, Equation (1) can be rewritten as

$$\begin{cases} dY = [\triangle Y + F(Y)]dt + \epsilon Y dW_t, \\ Y(0,x) = (u_0(x), v_0(x))^T. \end{cases} \tag{5}$$

Lemma 4. *For any Heaviside functions u_0 and v_0 as initial data, for, a.e., $\omega \in \Omega$, there exists a unique solution to Equation (5) in law with the form*

$$\begin{aligned} Y(t,x) = &\int_R G(t,x,y) Y_0 dy \\ &+ \int_0^t \int_R G(t-s,x,y) F(Y) ds dy + \epsilon \int_0^t \int_R G(t-s,x,y) Y dW_s dy, \end{aligned} \tag{6}$$

where $G(t,x,y)$ is the Green function.

Proof. We denote by $Y^n = (u^n, v^n)^T$, since $c_1(u,v)$ and $c_2(u,v)$ are Lipschitz continuous, perform some truncations and let $F_n(Y^n) = ((u^n \wedge \sqrt{n})c_1(u^n \wedge \sqrt{n}, v^n \wedge \sqrt{n}), (v^n \wedge \sqrt{n})c_2(u^n \wedge$

$\sqrt{n}, v^n \wedge \sqrt{n})$); then, $F_n(Y^n)$ is Lipschitz continuous. Hereto, the truncated Equation (7) can be constructed:

$$\begin{cases} dY^n = [\triangle Y^n + F_n(Y^n)]dt + \epsilon Y^n dW_t, \\ Y^n(0,x) = Y_0^n(x), \end{cases} \tag{7}$$

where $Y_0^n(x) \in C_{tem}^+$, and $Y_0^n(x) \to Y_0(x)$ as $n \to +\infty$. We refer to [12]. There exists a unique solution $Y^n(t,x)$ to Equation (7) in law and $Y^n(t,x) \in C_{tem}^+$. According to the Kolmogorov tightness criterion, we can show that for, a.e., $\omega \in \Omega$ there is a unique solution $Y(t,x)$ to Equation (5) such that $Y^n(t,x)$ converges to $Y(t,x)$ as $n \to \infty$. □

Lemma 5 ([12])**.** *All solutions to* (5) *with initial date* Y_0 *have the same law which is denoted by* $Q^{Y_0, \frac{\partial c_1}{\partial u}, \frac{\partial c_1}{\partial v}, \frac{\partial c_2}{\partial u}, \frac{\partial c_2}{\partial v}}$*, and map* $(Y_0, \frac{\partial c_1}{\partial u}, \frac{\partial c_1}{\partial v}, \frac{\partial c_2}{\partial u}, \frac{\partial c_2}{\partial v}) \to Q^{Y_0, \frac{\partial c_1}{\partial u}, \frac{\partial c_1}{\partial v}, \frac{\partial c_2}{\partial u}, \frac{\partial c_2}{\partial v}}$ *is continuous. For any Heaviside function* Y_0*, law* $Q^{Y_0, \frac{\partial c_1}{\partial u}, \frac{\partial c_1}{\partial v}, \frac{\partial c_2}{\partial u}, \frac{\partial c_2}{\partial v}}$ *forms a strong Markov family.*

Next, we perform several estimations about $Y(t,x)$ which play an important role in constructing travelling wave solution and estimating its asymptotic wave speed.

Theorem 1. *For any Heaviside functions* u_0 *and* v_0 *as initial data, if (H1)–(H4) hold, for any* $t > 0$ *fixed and, a.e.,* $\omega \in \Omega$*, it is true that*

$$\mathbb{E}[u(t,x) + v(t,x)] \leq C(\epsilon, t), \ \forall x \in R, \tag{8}$$

where $C(\epsilon,t) = \mathbb{E}[\exp(\sup\limits_{0 \leq r \leq t} \int_r^t \epsilon dW_s)]$ *is a constant.*

Proof. We consider Equation (1):

$$\begin{cases} du = [\triangle u + uc_1(u,v)]dt + \epsilon u dW_t, \\ dv = [\triangle v + vc_2(u,v)]dt + \epsilon v dW_t, \\ u(0,x) = u_0(x), v(0,x) = v_0(x). \end{cases} \tag{9}$$

We denote by $\phi(t,x) = u(t,x) + v(t,x)$; we have

$$\begin{cases} d\phi = [\triangle \phi + uc_1(u,v) + vc_2(u,v)]dt + \epsilon \phi dW_t, \\ \phi(0,x) = \phi_0 = u_0 + v_0. \end{cases} \tag{10}$$

We let $c_0 = \max\limits_{i=1,2}\{c_i(0,0)\}$; with (H1) and (H3), we know that there exists $\beta > a > 0$ such that

$$|c_1(u,v)| \leq c_0 + \beta|u| + a|v|, \ c_2(u,v) \leq c_0 + \beta|v| + a|u|, \tag{11}$$

combination with (H2) and (11) gives

$$c_1(u,v) \leq c_0 - \beta u + av, \ c_2(u,v) \leq c_0 - \beta v + au. \tag{12}$$

Frequently, it can be determined that

$$\begin{aligned} uc_1(u,v)+vc_2(u,v) \leq & u(c_0-\beta u+av)+v(c_0-\beta v+au) \\ \leq & c_0(u+v)-\beta(u^2+v^2)+2auv \\ \leq & c_0(u+v)-\frac{\beta-a}{2}(u+v)^2 \\ = & (u+v)(c_0-\frac{\beta-a}{2}(u+v)). \end{aligned}$$

We denote $c_3(x)=c_0-\frac{\beta-a}{2}x$; $c_3(x)$ is decreasing for $x\in[0,+\infty)$, and $c_3(x)<0$ for $x>\frac{2c_0}{\beta-a}$. We let $\psi(t,x)$ be the solution to Equation (13):

$$\begin{cases} d\psi=[\triangle\psi+\psi(c_0-\frac{\beta-a}{2}\psi)]dt+\epsilon\psi dW_t, \\ \psi_0=u_0+v_0. \end{cases} \tag{13}$$

According to monotone random dynamical systems theory [13] and its corresponding comparison principle [14], it can be determined that $u(t,x)\leq\psi(t,x)$ *a.s.* and $v(t,x)\leq\psi(t,x)$ *a.s.* We let $\zeta(t,x)$ be the solution to the following equation:

$$\begin{cases} d\zeta=[\triangle\zeta+\zeta(c_0-\frac{\beta-a}{2}\zeta)]dt-\frac{\epsilon^2}{2}\zeta, \\ \zeta_0=\psi_0. \end{cases} \tag{14}$$

We refer to [12] and use a stochastic Feynman–Kac formula. We have

$$\exp(\inf_{0\leq r\leq t}\int_r^t\epsilon dW_s)\zeta(t,x)\leq\psi(t,x)\leq\exp(\sup_{0\leq r\leq t}\int_r^t\epsilon dW_s)\zeta(t,x)\ a.s. \tag{15}$$

For any fixed $t>0$, for any $\sigma>0$, multiplying $G(t-s+\sigma,x-y)$ in (14) and integrating over R, it can be determined that

$$\begin{aligned} \frac{\partial}{\partial s}\int_R\zeta(s,y)G(t-s+\sigma,x-y)dy = & (c_0-\frac{\epsilon^2}{2})\int_R\zeta(s,y)G(t-s+\sigma,x-y)dy \\ & -\frac{\beta-a}{2}\int_R\zeta^2(s,y)G(t-s+\sigma,x-y)dy \\ \leq & (c_0-\frac{\epsilon^2}{2})\int_R\zeta(s,y)G(t-s+\sigma,x-y)dy \\ & -\frac{\beta-a}{2}(\int_R\zeta(s,y)G(t-s+\sigma,x-y)dy)^2. \end{aligned}$$

We let $\varphi(s)=\int_R\zeta(s,y)G(t-s+\sigma,x-y)dy$; then, we have

$$\begin{cases} \frac{d\varphi(s)}{ds}\leq(c_0-\frac{\epsilon^2}{2})\varphi(s)-\frac{\beta-a}{2}\varphi^2(s), \\ \varphi_0=\int_R\zeta_0G(t+\sigma,x-y)dy. \end{cases} \tag{16}$$

It can be deduced that

$$\varphi(s)\leq\varphi_0+\frac{2c_0}{\beta-a}-\frac{\epsilon^2}{\beta-a}. \tag{17}$$

Furthermore, we have

$$\int_R\zeta(t,y)G(\sigma,x-y)dy\leq\int_R\zeta_0G(t+\sigma,x-y)dy+\frac{2c_0}{\beta-a}-\frac{\epsilon^2}{\beta-a}. \tag{18}$$

Then, we let $\sigma \to 0$, and we have

$$\zeta(t,x) \leq \int_R \zeta_0 G(t+, x-y)dy + \frac{2c_0}{\beta - a} - \frac{\epsilon^2}{\beta - a}. \tag{19}$$

Therefore, we perform combination with (15) and obtain

$$\begin{aligned} u(t,x) + v(t,x) \leq \psi(t,x) \leq & \exp(\sup_{0 \leq r \leq t} \int_r^t \epsilon dW_s) \\ & \times (\int_R \psi_0 G(t, x-y)dy + \frac{2c_0}{\beta - a} - \frac{\epsilon^2}{\beta - a}) \ a.s. \end{aligned} \tag{20}$$

Moreover, since the initial data u_0 and v_0 are both Heaviside functions, we take the expectation and obtain

$$\mathbb{E}[u(t,x) + v(t,x)] \leq C(\epsilon, t)(u_0 + v_0 + \frac{2c_0}{\beta - a} - \frac{\epsilon^2}{\beta - a}), \tag{21}$$

where $C(\epsilon, t) = \mathbb{E}[\exp(\sup_{0 \leq r \leq t} \int_r^t \epsilon dW_s)]$. □

Theorem 2. *For any Heaviside functions u_0 and v_0 as initial data, if (H1)–(H4) hold, for, a.e., $\omega \in \Omega$, any $T > 0$ fixed, it it true that*

$$\mathbb{E} \sup_{0 \leq t \leq T} [|u(t)|^2 + |v(t)|^2] \leq \mathbb{E}[|u_0|^2 + |v_0|^2]e^{-t} + K(\epsilon)(1 - e^{-t}), \tag{22}$$

where $K(\epsilon) > 0$ is constant.

Proof. We let $V(t) = |u(t)|^2 + |v(t)|^2$ using the Itô formula, and we obtain

$$\begin{aligned} dV(t) =& 2\langle u, \triangle u \rangle dt + 2\langle v, \triangle v \rangle dt + 2\langle u, c_1(u,v) \rangle \, dt \\ & + 2\langle v, c_2(u,v) \rangle dt + \epsilon^2[u^2 + v^2]dt + 2\epsilon[u^2 + v^2]dW_t. \end{aligned}$$

Then, integrating both sides in $[0, t]$ and taking the expectation implies

$$\begin{aligned} \mathbb{E}[V(t)] =& \mathbb{E}[|u_0|^2 + |v_0|^2] + 2\mathbb{E}\int_0^t \langle u, \triangle u \rangle ds + 2\mathbb{E}\int_0^t \langle v, \triangle v \rangle ds + \epsilon^2 \mathbb{E}\int_0^t (u^2 + v^2)ds \\ & + 2\mathbb{E}\int_0^t \langle u, c_1(u,v) \rangle ds + 2\mathbb{E}\int_0^t \langle v, c_2(u,v) \rangle \, ds \\ \leq & \mathbb{E}[|u_0|^2 + |v_0|^2] - 2\mathbb{E}\int_0^t |\nabla u|^2 ds - 2\mathbb{E}\int_0^t |\nabla v|^2 ds + \epsilon^2 \mathbb{E}\int_0^t (u^2 + v^2)ds \\ & + 2c_0 \mathbb{E}\int_0^t (u^2 + v^2)ds - 2(\beta - a)\mathbb{E}\int_0^t (u^3 + v^3)ds \\ \leq & \mathbb{E}[|u_0|^2 + |v_0|^2] - 2(\beta - a)\mathbb{E}\int_0^t (u^3 + v^3)ds + 2c_0 \mathbb{E}\int_0^t (u^2 + v^2)ds \\ & + \epsilon^2 \mathbb{E}\int_0^t (u^2 + v^2)ds + \mathbb{E}\int_0^t (u^2 + v^2)ds - \mathbb{E}\int_0^t (u^2 + v^2)ds. \end{aligned}$$

Hence, with Young inequality and Gronwall inequality, we have

$$\mathbb{E}[|u(t)|^2 + |v(t)|^2] \leq \mathbb{E}[|u_0|^2 + |v_0|^2]e^{-t} + K(\epsilon)(1 - e^{-t}), \tag{23}$$

where $K(\epsilon)$ is a constant. □

With the boundedness of $Y(t,x)$, the sup-solution can be constructed to describe how fast the support of $Y(t,x)$ can spread and the SCP property can be obtained.

Lemma 6. *We let $Y(t,x)$ be the solution to (1) with initial data Y_0 as a Heaviside function. We suppose for some $r>0$ such that Y_0 is supported outside $(-r-2, r+2)$; then, for any $t \geq 1$,*

$$\begin{aligned}&\mathbb{P}(\int_0^t \int_{-r}^{r} ||Y(s,x)||_\infty dsdx > 0)\\ &\qquad\qquad \leq Ce^t \int \frac{\sqrt{t}}{|x|-(r+1)} \exp(-\frac{(|x|-(r+1))^2}{2t})||Y_0||_\infty dx.\end{aligned} \tag{24}$$

Proof. Since $Y(t,x)$ is bounded and we construct a sup-solution solving the following equation:

$$\begin{cases} du^\star = [\triangle u^\star + u^\star(\rho - \beta u^\star)]dt + \epsilon u^\star dW_t, \\ dv^\star = [\triangle v^\star + v^\star(\rho - \beta v^\star)]dt + \epsilon v^\star dW_t, \\ u^\star(0) = u_0, v^\star(0) = v_0, \end{cases} \tag{25}$$

where $\rho > 0$ is a constant such that $uc_1(u,v) \leq u(\rho - \beta u)$ and $vc_2(u,v) \leq v(\rho - \beta v)$, then we refer to [11,12]. The proof can be completed. $\square$

Then, we show that $Y(t,x)$ satisfy the Kolmogorov tightness criterion and $Y(t,x) \in K(C,\delta,\mu,\gamma)$, which is dedicated to constructing a tight probability measure sequence and furthermore obtain the existence of a travelling wave solution.

Lemma 7. *For any Heaviside functions u_0 and v_0 as initial data, if (H1)–(H4) hold, for, a.e., $\omega \in \Omega$, any $p \geq 2$ fixed, $t > 0$, if $|x-x'| \leq 1$, there exits $C(p,t) < \infty$ such that*

$$Q^{Y_0}(|Y(t,x) - Y(t,x')|^p) \leq C(p,t)|x-x'|^{p/2-1}.$$

Proof. Direct calculation shows that

$$\begin{aligned}&|Y(t,x) - Y(t,x')|^p\\ &\quad \leq 3^{p-1}|\int_R (G(t,x-y) - G(t,x'-y))u_0 dy|^p\\ &\qquad + 3^{p-1}|\int_R (G(t,x-y) - G(t,x'-y))v_0 dy|^p\\ &\qquad + \underbrace{3^{p-1}|\int_R\int_0^t (G(t-s,x-y) - G(t-s,x'-y))uc_1(u,v)dsdy|^p}_{I}\end{aligned}$$

$$\begin{aligned}&+ \underbrace{3^{p-1}|\int_R\int_0^t (G(t-s,x-y) - G(t-s,x'-y))vc_2(u,v)dsdy|^p}_{II}\\ &+ \underbrace{3^{p-1}\epsilon^p|\int_R\int_0^t (G(t-s,x-y) - G(t-s,x'-y))udW_s dy|^p}_{III}\\ &+ \underbrace{3^{p-1}\epsilon^p|\int_R\int_0^t (G(t-s,x-y) - G(t-s,x'-y))vdW_s dy|^p}_{IV}.\end{aligned}$$

Since

$$\int_0^t \int_R (G(t-s,x-y) - G(t-s,x'-y))^2 dsdy \leq C(t)|x-x'|,$$

and $Y(t,x)$ is bounded, for III and IV, we have

$$\begin{aligned}\mathbb{E}[III] \leq & C(p)\epsilon^p \mathbb{E}\Big(\int_0^t \int_R (G(t-s,x-y)-G(t-s,x'-y))^2 dsdy\Big)^{p/2-1} \\ & \times \Big(\int_0^t \int_R (G(t-s,x-y)-G(t-s,x'-y))^2 u^p dsdy\Big) \\ \leq & C_1(p,t)|x-x'|^{p/2-1},\end{aligned}$$

and

$$\mathbb{E}[IV] \leq C_2(p,t)|x-x'|^{p/2-1}.$$

With Hölder inequality, for I and II, we have

$$\begin{aligned}\mathbb{E}[I] = & 3^{p-1}\mathbb{E}\Big|\int_0^t \int_R (G(t-s,x-y)-G(t-s,x'-y))(u-a_1u^2+b_1uv)dsdy\Big|^p \\ \leq & 3^{p-1}\Big(\int_0^t \int_R (G(t-s,x-y)-G(t-s,x'-y))^2 dsdy\Big)^{p/2-1} \\ & \times \Big(\int_0^t \int_R |G(t-s,x-y)-G(t-s,x'-y)|^2 \mathbb{E}[(u-a_1u^2+b_1uv)^p]dsdy\Big) \\ \leq & C_3(p,t)|x-x'|^{p/2-1},\end{aligned}$$

and

$$\mathbb{E}[II] \leq C_4(p,t)|x-x'|^{p/2-1}.$$

Meanwhile, we have

$$\begin{aligned}& \mathbb{E}\Big|\int_R (G(t,x-y)-G(t,x'-y))u_0 dy\Big|^p \\ & = \mathbb{E}\Big|\int_R \int_{x'}^x \frac{(y-r)}{2t\sqrt{4\pi y}} \exp\Big(-\frac{(y-r)^2}{4t}\Big)u_0 drdy\Big|^p \\ & \leq K(t)\Big(\int_R \int_{x'}^x \frac{1}{\sqrt{t}} \exp\Big(-\frac{(y-r)^2}{5t}\Big)u_0 drdy\Big)^p \\ & \leq K(t)|x-x'|^p \int_R \frac{1}{\sqrt{t}} \exp\Big(-\frac{(y-x)^2}{5t}\Big)|u_0|^p dy \\ & \leq C_5(p,t)|x-x'|^{p/2-1}, \quad (since\ |x-x'| \leq 1),\end{aligned}$$

and

$$\mathbb{E}\Big|\int_R (G(t,x-y)-G(t,x'-y))v_0 dy\Big|^2 \leq C_6(p,t)|x-x'|^{p/2-1}.$$

Combining all the inequalities above completes the proof:

$$\mathbb{E}[|Y(t,x)-Y(t,x')|^p] \leq C(p,t)|x-x'|^{p/2-1}. \tag{26}$$

□

In order to construct a travelling weave solution, according to the two sufficient conditions proposed by Tribe [11], we are required to show that the wavefront marker is bounded for all $t > 0$ and the translation of solution with respect to a wavefront marker is stationary. However, it is quite difficult to deal with $R_0(t)$ directly, so we turn to a new suitable wavefront marker for help. We define Q^{Y_0} as the law of the unique solution to Equation (5) with initial data Y_0. For a probability measure ν on C^+_{tem}, we define

$$Q^{\nu}(A) = \int_{C^+_{tem}} Q^{Y_0}(A)\nu(dY_0),$$

define a new wavefront marker $R_1(t) : C^+_{tem} \to [-\infty, \infty]$,

$$R_1(f) = ln \int_R e^x f dx,\ R_1(u(t)) = \ln \int_R e^x u(t,x)dx,$$

and

$$R_1(t) := R_1(Y(t)) = \max\{R_1(u(t)), R_1(v(t))\};$$

then, $R_1(t)$ is an approximation to $R_0(Y(t)) = \max\{R_0(u(t)), R_0(v(t))\}$. We let $Z(t) = Y(t, \cdot + R_1(t)) = (Z_1(t), Z_2(t))^T$, $Z_0(t) = Y(t, \cdot + R_0(Y(t)))$, and define

$$Z(t) = \begin{cases} (0,0)^T, & R_1(t) = -\infty, \\ (u(t, \cdot + R_1(t)), v(t, \cdot + R_1(t)))^T, & -\infty < R_1(t) < \infty, \\ (p_1, p_2)^T, & R_1(t) = \infty. \end{cases}$$

Next, define

$$\nu_T = \textit{ the law of } \frac{1}{T}\int_0^T Z(s)ds \textit{ under } Q^{Y_0}.$$

Now, we summarize the steps of constructing a travelling wave solution. First, we show that the new wavefront marker is bounded (see Lemma 8) to ensure the shifting does not destroy the tightness (see Lemma 7). Based on this, we construct a tight probability measure sequence $\{\nu_T\}_{T\in\mathbb{N}}$ (see Lemma 9) and show that any limit point is nontrivial (see Theorem 3), where Q^{ν} is the law of a travelling wave solution.

Lemma 8. *For any Heaviside functions u_0 and v_0 as initial data, for, a.e., $\omega \in \Omega$, any $t \geq 0$, $d > 0$, $T \geq 1$, there exists $C(t) < \infty$ such that*

$$Q^{\nu_T}(|R_1(t)| > d) \leq \frac{C(t)}{d}. \tag{27}$$

Proof. With the comparison principle, we construct a sup-solution solving Equation (28):

$$\begin{cases} d\tilde{u} = [\triangle \tilde{u} + k\tilde{u}]dt + \epsilon \tilde{u} dW_t, \\ d\tilde{v} = [\triangle \tilde{v} + k\tilde{v}]dt + \epsilon \tilde{v} dW_t, \\ \tilde{u}_0 = u_0, \tilde{v}_0 = v_0, \end{cases} \tag{28}$$

where $k > 0$ is a constant such that $uc_1(u,v) \leq ku$ and $vc_2(u,v) \leq kv$; thus, we determine that $u(t,x) \leq \tilde{u}(t,x)$ and $v(t,x) \leq \tilde{v}(t,x)$ hold on $[0,T]$ uniformly, and for, a.e., $\omega \in \Omega$ the solution to Equation (28) can be expressed by

$$\tilde{Y}(t,x) = \int_R e^{kt} G(t, x-y) Y_0(y) dy + \epsilon \int_R \int_0^t G(t-s, x-y) \tilde{Y} dW_s dy. \tag{29}$$

Without generality, we assume that $R_1(t) = R_1(u(t))$ and take $u(t,x)$ for an example. We have

$$\begin{aligned} Q^{u_0}(\int_R u(t,x)e^x dx) \leq & \mathbb{E}[\int_R \tilde{u}(t,x)e^x dx] \\ = & \mathbb{E}[\int_R \int_R e^{kt} G(t, x-y)u_0(y)dye^x dx] \\ = & e^{kt+t} \int_R u_0(x)e^x dx, \end{aligned} \tag{30}$$

according to the definition of $R_1(t)$, we know

$$\int_R u(t, x + R_1(t))e^x dx = e^{-R_1(t)} \int_R u(t,x)e^x dx = 1; \tag{31}$$

meanwhile, we have

$$\int_R v(t, x + R_1(t))e^x dx \leq 1.$$

Combining (30) with (31) implies that

$$\begin{aligned} Q^{v_T}(R_1(t) \geq d) = & \frac{1}{T} \int_0^T Q^{u_0}(Q^{u(s)}(R_1(t) \geq d))ds \\ = & \frac{1}{T} \int_0^T Q^{u_0}(Q^{u(s)}(e^{-d} \int_R u(t,x)e^x dx \geq 1))ds \\ \leq & e^{-d} \frac{1}{T} \int_0^T Q^{u_0}(Q^{u(s)}(\int_R u(t,x)e^x dx))ds \\ \leq & e^{-d} e^{kt+t} \frac{1}{T} \int_0^T \int_R u(s, x + R_1(s))e^x dxds \\ = & e^{-d} e^{kt+t}. \end{aligned} \tag{32}$$

On the other hand, Jensen's inequality offers

$$Q^{u_0}(R_1(t)) \leq \ln(e^{kt+t} \int_R u_0(x)e^x dx) \leq kt + t + R_1(u_0),$$

additionally, we can obtain such estimation:

$$\begin{aligned} & \frac{1}{T} Q^{u_0}(\int_t^{T+t} R_1(s)ds - \int_0^T R_1(s)ds) \\ = & \frac{1}{T} Q^{u_0}(\int_0^T R_1(t+s) - R_1(s)ds) \\ = & \frac{1}{T} \int_0^T \int_{\{R_1(t+s)-R_1(s)>-d\}} (R_1(t+s) - R_1(s))Q^{u_0}(du)ds \\ & + \frac{1}{T} \int_0^T \int_{\{R_1(t+s)-R_1(s)\leq -d\}} (R_1(t+s) - R_1(s))Q^{u_0}(du)ds \\ \leq & \frac{1}{T} \int_0^T \int_{\{R_1(t+s)-R_1(s)>0\}} (R_1(t+s) - R_1(s))Q^{u_0}(du)ds \\ & - \frac{d}{T} \int_0^T Q^{u_0}(R_1(t+s) - R_1(s) \leq -d)ds \\ \leq & \frac{1}{T} \int_0^T \int_0^\infty Q^{u_0}(R_1(t+s) - R_1(s) \geq y)dyds \end{aligned}$$

$$
\begin{aligned}
&-\frac{d}{T}\int_0^T Q^{u_0}(R_1(t+s)-R_1(s)\le -d)ds\\
=&\int_0^\infty Q^{\nu_T}(R_1(t)\ge y)dy - dQ^{\nu_T}(R_1(t)\le -d).
\end{aligned}
$$

Rearranging the inequalities above implies

$$
\begin{aligned}
Q^{\nu_T}(R_1(t)\le -d)\le&\frac{1}{d}\int_0^\infty Q^{\nu_T}(R_1(t)\ge y)dy+\frac{1}{dT}\int_0^T Q^{\nu_T}(R_1(s))ds\\
&-\frac{1}{dT}\int_t^{T+t} Q^{u_0}(R_1(s))ds\\
\le&\frac{1}{d}\int_0^\infty e^{-y+k_0t+t}dy+\frac{1}{dT}\int_0^T k_0s+s+R_1(u_0)ds\\
\le&\frac{C(t)}{d}.
\end{aligned}
\tag{33}
$$

We combine (32) with (33), and the proof can be completed. □

Via the boundedness of wavefront marker $R_1(t)$, we can construct the tight sequence $\{\nu_T : T\in\mathbb{N}\}$ with $Y(t,x)\in K(C,\delta,\mu,\gamma)$.

Lemma 9. *For any Heaviside functions u_0 and v_0 as initial data, for, a.e., $\omega\in\Omega$, the sequence $\{\nu_T : T\in\mathbb{N}\}$ is tight.*

Proof. Similarly, we start discussion with $u(t,x)$. Since $Y(t,x)\in K(C,\delta,\mu,\gamma)$, it can be determined that $u(t,x)\in K(C,\delta,\mu,\gamma)$, and furthermore, we have

$$
\begin{aligned}
\nu_T(K(C,\delta,\gamma,\mu))=&\frac{1}{T}\int_0^T Q^{u_0}(u(t,\cdot+R_1(t))\in K(C,\delta,\gamma,\mu))ds\\
\ge&\frac{1}{T}\int_0^T Q^{u_0}((u(t,\cdot+R_1(t-1))\in K(Ce^{-\mu d},\delta,\gamma,\mu))\\
&\times|R_1(t)-R_1(t-1)|\le d)ds\\
:=&I-II.
\end{aligned}
$$

According to Lemma 8, it can be easily determined that $II\to 0$ as $d\to\infty$. In addition, for any given $d,\mu>0$, we choose C,δ,γ to make I as close to $\frac{T}{T-1}$ as desired. In addition, we know

$$
\nu_T\{u_0:\int_R u_0(x)e^{-|x|}dx\le\int_R u_0(x)e^x dx=1\}=1;
$$

thus, for a given $\mu>0$, we can choose C,δ,γ such that $\nu_T(K(C,\delta,\mu,\gamma)\cap\{u_0:\int_R u_0(x)e^{-|x|}dx\})$ as close to one as desired for T and d that are sufficiently large, which means that sequence $\{\nu_T : T\in\mathbb{N}\}$ is tight. We refer to Lemma 3.9 in [12]. The proof can be completed. □

Theorem 3. *For any Heaviside functions u_0 and v_0 as initial data, if (H1)–(H4) hold, for, a.e., $\omega\in\Omega$, any $T>0$ fixed, there is a travelling wave solution to Equation (1), and Q^ν is the law of travelling wave solution.*

Proof. Refer to Theorem 3.10 in [12]. The proof can be completed. □

Based on the existence of a stochastic travelling wave solution, we can estimate the wave speed by the comparison principle in the following. First, we construct the sup-solution

$$\begin{cases} d\bar{u} = [\triangle\bar{u} + \bar{u}\bar{c}(\bar{u})]dt + \epsilon\bar{u}dW_t, \\ d\bar{v} = [\triangle\bar{v} + \bar{v}\bar{c}(\bar{v})]dt + \epsilon\bar{v}dW_t, \\ \bar{u}_0 = \frac{\bar{c}(0)}{c_1(0,0)}u_0, \bar{v}_0 = \frac{\bar{c}(0)}{c_2(0,0)}v_0, \end{cases} \tag{34}$$

where $\bar{c}(x)$ is Lipschitz continuous and decreasing for $x \in [0, \infty)$, and there exists $\delta > 0$, such that $\bar{c}(x) < 0$ for $x > \delta$. Since (u, v) is the solution to (1) and bounded, it holds that $c_1(u, v) \leq \bar{c}(u)$ and $c_2(u, v) \leq \bar{c}(v)$. Then, we construct the sub-solution

$$\begin{cases} d\underline{u} = [\triangle\underline{u} + \underline{u}c_1(\underline{u}, 0)]dt + \epsilon\underline{u}dW_t, \\ d\underline{v} = [\triangle\underline{v} + \underline{v}c_2(0, \underline{v})]dt + \epsilon\underline{v}dW_t, \\ \underline{u}_0 = u_0, \underline{v}_0 = v_0, \end{cases} \tag{35}$$

where $c_1(u, v) \geq c_1(u, 0), c_2(u, v) \geq c_2(0, v)$. According to the comparison theorem of wave speed (see Lemma 4.2 in [12]), the wave speed of travelling wave solution to Equation (1) can be estimated.

Theorem 4. *For any Heaviside functions u_0 and v_0 as initial data, if (H1)–(H4) hold, we denote by c the wave speed of the travelling wave solution to Equation (1); then,*

$$\sqrt{4\underline{c}_0 - 2\epsilon^2} \leq c \leq \sqrt{4\bar{c}_0 - 2\epsilon^2} \quad a.s. \tag{36}$$

where $\underline{c}_0 = \max\{c_1(0, 0), c_2(0, 0)\}$, $\bar{c}_0 = \bar{c}(0)$.

Proof. According the definition of wave speed,

$$c = \lim_{t\to\infty} \frac{R_0(t)}{t} \; a.s.,$$

the wave speed of stochastic reaction–diffusion equations is the maximum value between the two sub-systems; thus, with the comparison theorem of wave speed and referring to Theorems 4.1 and 4.2 in [12], the proof can be completed. □

3. Conclusions

In this paper, we are devoted to the propagation dynamics of stochastic reaction–diffusion equations and offer the definition of the stochastic travelling wave solution in law. According to the random monotonic dynamical system theory, the existence of travelling wave solution is determined via the two sufficient conditions proposed by Tribe, and we summarize the general methods of constructing a stochastic travelling wave solution. Furthermore, the estimation of asymptotic wave speed can be obtained by constructing sup-solution and sub-solution and using the comparison method of wave speed. Obviously, the upper bound and the lower bound of wave speed depend on the nonlinear terms and the strength of noise, and it is in line with reality. In the deterministic condition, the wave speed relies on the nonlinear term. After introducing the noise term, its impact is reflected in the estimate of wave speed.

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