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Existence of Solutions to a System of Fractional q-Difference Boundary Value Problems

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Abstract: We are investigating the existence of solutions to a system of two fractional $\mathfrak{q}$-difference equations containing fractional $\mathfrak{q}$-integral terms, subject to multi-point boundary conditions that encompass $\mathfrak{q}$-derivatives and fractional $\mathfrak{q}$-derivatives of different orders. In our main results, we rely on various fixed point theorems, such as the Leray–Schauder nonlinear alternative, the Schaefer fixed point theorem, the Krasnosel'skii fixed point theorem for the sum of two operators, and the Banach contraction mapping principle. Finally, several examples are provided to illustrate our findings.

Keywords: fractional $\mathfrak{q}$-difference equations; coupled multi-point boundary conditions; fractional $\mathfrak{q}$-integrals; existence; uniqueness

MSC: 39A13; 39A27; 33D05

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1. Introduction

We examine the system of fractional $\mathfrak{q}$-difference equations

$$\begin{cases} (\mathrm{D}_{\mathfrak{q}}^{\alpha}\mathrm{u})(\nu) + \mathcal{F}\Big(\nu, \mathrm{u}(\nu), \mathrm{v}(\nu), \mathrm{I}_{\mathfrak{q}}^{\delta_1}\mathrm{u}(\nu), \mathrm{I}_{\mathfrak{q}}^{\gamma_1}\mathrm{v}(\nu)\Big) = 0, \quad \nu \in (0,1), \\ \Big(\mathrm{D}_{\mathfrak{q}}^{\beta}\mathrm{v}\Big)(\nu) + \mathcal{G}\Big(\nu, \mathrm{u}(\nu), \mathrm{v}(\nu), \mathrm{I}_{\mathfrak{q}}^{\delta_2}\mathrm{u}(\nu), \mathrm{I}_{\mathfrak{q}}^{\gamma_2}\mathrm{v}(\nu)\Big) = 0, \quad \nu \in (0,1), \end{cases} \tag{1}$$

subject to the multi-point boundary conditions

$$\begin{cases} \mathrm{D}_{\mathfrak{q}}^{i}\mathrm{u}(0) = 0, \;\; i = 0,\ldots,n-2, \;\; \mathrm{D}_{\mathfrak{q}}^{\varsigma}\mathrm{u}(1) = \sum\limits_{i=1}^{\mathfrak{a}} a_i \mathrm{D}_{\mathfrak{q}}^{\varrho_i}\mathrm{u}(\xi_i) + \sum\limits_{i=1}^{\mathfrak{b}} b_i \mathrm{D}_{\mathfrak{q}}^{\sigma_i}\mathrm{v}(\omega_i), \\ \mathrm{D}_{\mathfrak{q}}^{i}\mathrm{v}(0) = 0, \;\; i = 0,\ldots,m-2, \;\; \mathrm{D}_{\mathfrak{q}}^{\vartheta}\mathrm{v}(1) = \sum\limits_{i=1}^{\mathfrak{c}} c_i \mathrm{D}_{\mathfrak{q}}^{\eta_i}\mathrm{u}(\zeta_i) + \sum\limits_{i=1}^{\mathfrak{d}} d_i \mathrm{D}_{\mathfrak{q}}^{\rho_i}\mathrm{v}(\theta_i). \end{cases} \tag{2}$$

Here, $\mathfrak{q} \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \geq 2$; $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathbb{N}$; $\varsigma, \varrho_i, \eta_k \in [0, \alpha-1)$; $\vartheta, \sigma_j, \rho_\iota \in [0, \beta-1)$; $a_i, b_j, c_k, d_\iota \in \mathbb{R}$; $\xi_i, \omega_j, \zeta_k, \theta_\iota \in (0,1)$; $\mathrm{D}_{\mathfrak{q}}^{\kappa}$ is the fractional $\mathfrak{q}$-derivative of order κ, for $\kappa = \alpha, \beta, \varsigma, \vartheta, \varrho_i, \sigma_j, \eta_k, \rho_\iota$, for all $i = 1,\ldots,\mathfrak{a}$, $j = 1,\ldots,\mathfrak{b}$, $k = 1,\ldots,\mathfrak{c}$, $\iota = 1,\ldots,\mathfrak{d}$; $\mathrm{D}_{\mathfrak{q}}^{p}$ represents the $\mathfrak{q}$-derivative of order p, for $p = 0,\ldots,n-2$ and $p = 0,\ldots,m-2$; $\delta_r, \gamma_r > 0$ for $r = 1,2$; $\mathrm{I}_{\mathfrak{q}}^{\kappa}$ is the fractional $\mathfrak{q}$-integral of order κ, for $\kappa = \delta_i, \gamma_i$, $i = 1,2$, and $\mathcal{F}, \mathcal{G}$ are nonlinear functions satisfying some assumptions.

In this paper, we aim to set forth conditions on the functions $\mathcal{F}$ and $\mathcal{G}$ that guarantee the existence of at least one solution to problem (1), (2). Our proofs will make use of various fixed-point theorems, including the Leray–Schauder nonlinear alternative, the Schaefer fixed-point theorem, the Krasnosel'skii fixed-point theorem for the sum of two operators, and the Banach contraction mapping principle. Furthermore, we will include references to relevant literature closely associated with our investigated problem. In [1], the author studied the existence, uniqueness, and multiplicity of positive solutions for problem (1), (2) under different assumptions than those used in our present paper. The associated Green functions are constructed, and some of their properties are presented. For the proof of the principal findings, the author employed in [1] a range of fixed point theorems, including the Schauder fixed point theorem, the Leggett–Williams fixed point theorem, and the Guo–Krasnosel'skii fixed point theorem. Therefore, the methods used in [1] are distinct from those we will apply in our paper. In [2], the authors investigated the system of nonlinear fractional $\mathfrak{q}$-difference equations

$$\begin{cases} (D_q^{\alpha_1}u)(t) + P(t, u(t), v(t), I_q^{\omega_1}u(t), I_q^{\delta_1}v(t)) = 0, \ \ t \in (0,1), \\ (D_q^{\alpha_2}v)(t) + Q(t, u(t), v(t), I_q^{\omega_2}u(t), I_q^{\delta_2}v(t)) = 0, \ \ t \in (0,1), \end{cases} \tag{3}$$

with the coupled nonlocal boundary conditions

$$\begin{cases} D_q^i u(0) = 0, \ \ i = 0, \dots, m-2, \ \ D_q^{\zeta_0}u(1) = \int_0^1 D_q^{\zeta}v(t)\, d_q H(t), \\ D_q^i v(0) = 0, \ \ i = 0, \dots, n-2, \ \ D_q^{\xi_0}v(1) = \int_0^1 D_q^{\xi}u(t)\, d_q K(t), \end{cases} \tag{4}$$

where $\mathfrak{q} \in (0,1)$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \in (m-1, m]$, $\alpha_2 \in (n-1, n]$, $m, n \in \mathbb{N}$, $m \geq 2$, $n \geq 2$, $\omega_i > 0$, $\delta_i > 0$, $i = 1,2$, $\zeta \in [0, \alpha_2 - 1)$, $\xi \in [0, \alpha_1 - 1)$, $\zeta_0 \in [0, \alpha_1 - 1)$, $\xi_0 \in [0, \alpha_2 - 1)$, the integrals from (4) are Riemann–Stieltjes integrals, and H and K are bounded variation functions. Utilizing diverse fixed-point theorems, they established results affirming the existence and uniqueness of solutions to problem (3), (4). In [3], the authors analyzed the existence of solutions to the fractional $\mathfrak{q}$-difference equation subject to nonlocal boundary conditions

$$\begin{cases} ({}^C D_q^{\beta}u)(t) = f(t, u(t)), \ \ \text{for a.e. } t \in (0,T), \\ u(0) - u'(0) = \int_0^T h(s, u(s))\, ds, \ \ u(T) + u'(T) = \int_0^T g(s, u(s))\, ds, \end{cases} \tag{5}$$

where $T > 0$, $\mathfrak{q} \in (0,1)$, $\beta \in (1,2]$, and ${}^C D_q^{\alpha}$ is the Caputo fractional $\mathfrak{q}$-derivative of order α. In demonstrating the main result, they employed the Mönch fixed-point theorem, and the method of measures of noncompactness. In [4], the authors examined the existence, uniqueness and multiplicity of positive solutions to the fractional $\mathfrak{q}$-difference equation supplemented with nonlocal boundary conditions

$$\begin{cases} (D_q^{\beta}u)(t) + g(t, u(t)) = 0, \ \ t \in (0,1), \\ (D_q^i u)(0) = 0, \ \ i = 0, \dots, m-2, \ \ (D_q^{\gamma}u)(1) = a(D_q^{\gamma}u)(\eta), \end{cases} \tag{6}$$

where $\mathfrak{q} \in (0,1)$, $\beta \in (m-1, m]$, $m > 2$, $\gamma \in [1, m-2]$, $\eta \in (0,1)$, $a \in [0,1]$, and $g : [0,1] \times [0,\infty) \to [0,\infty)$ satisfies Caratheodory type conditions. In proving the main theorems, they utilized multiple fixed-point theorems. In [5], based on the Guo–Krasnosel'skii fixed point theorem, the author explored the existence of positive solutions for the fractional $\mathfrak{q}$-difference equation subject to boundary conditions

$$\begin{cases} (D_q^{\gamma}v)(t) = -g(t, v(t)), \ \ t \in (0,1), \\ v(0) = (D_q v)(0) = 0, \ \ (D_q v)(1) = \beta, \end{cases} \tag{7}$$

where $\mathfrak{q} \in (0,1)$, $\gamma \in (2,3]$, $\beta \geq 0$, and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. In [6], the author studied the existence of nontrivial solutions for the nonlinear $\mathfrak{q}$-fractional boundary value problem

$$\begin{cases} (D_q^{\gamma}v)(t) = -g(t, v(t)), \ \ t \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{8}$$

where $\mathfrak{q} \in (0,1)$, $\gamma \in (1,2]$, and $\mathrm{g} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. To prove the main results, he also used the Guo–Krasnosel'skii fixed point theorem. For other research works that investigate fractional $\mathfrak{q}$-difference equations and systems of fractional $\mathfrak{q}$-difference equations with either coupled or uncoupled boundary conditions, we refer the reader to the following papers [7–14].

The domain of $\mathfrak{q}$-difference calculus, commonly known as quantum calculus, finds its roots in the seminal contributions of Jackson [15,16]. For a comprehensive exploration of diverse applications within this field, readers are encouraged to delve into the research conducted by Ernst [17]. The inception of fractional $\mathfrak{q}$-difference calculus can be traced back to the works of Al-Salam [18] and Agarwal [19]. To stay updated on advancements in this subfield, covering $\mathfrak{q}$-analogs of integral and differential fractional operators, along with properties such as $\mathfrak{q}$-analogs of Cauchy's formula, the fractional Leibniz $\mathfrak{q}$-formula, $\mathfrak{q}$-Taylor's formula, $\mathfrak{q}$-Laplace transform, and $\mathfrak{q}$-analogs of the Mittag–Leffler function, see the papers [19–31].

The novelty aspects of our problem (1), (2), compared to that examined in [1] are the following. In our paper, we study the existence of solutions for problem (1), (2), in contrast to [1], where the author investigated the existence of positive solutions for (1), (2). For this reason, the assumptions on the orders of the fractional derivatives in [1] are stronger than those used here, and they assure the nonnegativity of the associated Green functions. Indeed, in [1], the orders ς and ϑ must be greater than or equal to 1, an assumption that does not appear in our present work. In addition, in [1], there are connections between ς, ϱ_i and η_k for $i = 1, \ldots, \mathfrak{a}$ and $k = 1, \ldots, \mathfrak{c}$, on the one hand, and ϑ, σ_j and ρ_ι for $j = 1, \ldots, \mathfrak{b}$ and $\iota = 1, \ldots, \mathfrak{d}$, on the other hand. Namely, ϱ_i and η_k are less than or equal to ς, for $i = 1, \ldots, \mathfrak{a}$ and $k = 1, \ldots, \mathfrak{c}$, and σ_j and ρ_ι are less than or equal to ϑ, for $j = 1, \ldots, \mathfrak{b}$ and $\iota = 1, \ldots, \mathfrak{d}$. These last conditions are not used in our paper. Furthermore, the theorems applied in the present paper are different than those utilized in [1]. Related to paper [2], the differences between [2] and our paper are in the form of boundary conditions, which in our case (boundary conditions (2)) are more general than the conditions (4); our conditions (2) are generalized coupled boundary conditions.

Our paper is structured as follows: Section 2 presents auxiliary results essential for the subsequent sections. In Section 3, we unveil the primary existence results for the problem (1), (2). Moving on, Section 4 offers illustrative examples to showcase the applicability of our theorems. Finally, Section 5 concludes the paper by providing a summary of the findings and presenting comprehensive conclusions.

2. Auxiliary Results

This section provides initial findings that will be utilized in subsequent sections. We begin by examining the linear system associated with our given problem (1), (2), namely

$$\begin{cases} (D_{\mathfrak{q}}^{\alpha}\mathrm{u})(\nu) + h(\nu) = 0, \;\; \nu \in (0,1), \\ (D_{\mathfrak{q}}^{\beta}\mathrm{v})(\nu) + k(\nu) = 0, \;\; \nu \in (0,1), \end{cases} \tag{9}$$

with the boundary conditions (2), where $h,\, k \in C[0,1]$.

We introduce the constants

$$\begin{aligned} &\Lambda_1 = \frac{\Gamma_{\mathfrak{q}}(\alpha)}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)} - \sum_{i=1}^{\mathfrak{a}} a_i \frac{\Gamma_{\mathfrak{q}}(\alpha)}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)} \xi_i^{\alpha-\varrho_i-1}, \\ &\Lambda_2 = \sum_{i=1}^{\mathfrak{b}} b_i \frac{\Gamma_{\mathfrak{q}}(\beta)}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)} \omega_i^{\beta-\sigma_i-1}, \;\; \Lambda_3 = \sum_{i=1}^{\mathfrak{c}} c_i \frac{\Gamma_{\mathfrak{q}}(\alpha)}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)} \zeta_i^{\alpha-\eta_i-1}, \\ &\Lambda_4 = \frac{\Gamma_{\mathfrak{q}}(\beta)}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)} - \sum_{i=1}^{\mathfrak{d}} d_i \frac{\Gamma_{\mathfrak{q}}(\beta)}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)} \theta_i^{\beta-\rho_i-1}, \\ &\Delta = \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3. \end{aligned} \tag{10}$$

Lemma 1 ([1]). *If $\Delta \neq 0$, then the solution $(\mathrm{u}(\nu), \mathrm{v}(\nu))$, $\nu \in [0,1]$ of problem (9), (2) is given by*

$$
\begin{aligned}
\mathrm{u}(\nu) = & -\frac{1}{\Gamma_{\mathfrak{q}}(\alpha)} \int_0^{\nu} (\nu - \mathfrak{q}\tau)^{(\alpha-1)} h(\tau)\, d_{\mathfrak{q}}\tau \\
& + \frac{\nu^{\alpha-1}}{\Delta} \left[\frac{\Lambda_4}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)} \int_0^1 (1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)} h(\tau)\, d_{\mathfrak{q}}\tau \right. \\
& - \Lambda_4 \sum_{i=1}^{\mathfrak{a}} \frac{a_i}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)} \int_0^{\xi_i} (\xi_i - \mathfrak{q}\tau)^{(\alpha-\varrho_i-1)} h(\tau)\, d_{\mathfrak{q}}\tau \\
& \left. - \Lambda_2 \sum_{i=1}^{\mathfrak{c}} \frac{c_i}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)} \int_0^{\zeta_i} (\zeta_i - \mathfrak{q}\tau)^{(\alpha-\eta_i-1)} h(\tau)\, d_{\mathfrak{q}}\tau \right] \\
& + \frac{\nu^{\alpha-1}}{\Delta} \left[-\Lambda_4 \sum_{i=1}^{\mathfrak{b}} \frac{b_i}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)} \int_0^{\omega_i} (\omega_i - \mathfrak{q}\tau)^{(\beta-\sigma_i-1)} k(\tau)\, d_{\mathfrak{q}}\tau \right. \\
& + \frac{\Lambda_2}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)} \int_0^1 (1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)} k(\tau)\, d_{\mathfrak{q}}\tau \\
& \left. - \Lambda_2 \sum_{i=1}^{\mathfrak{d}} \frac{d_i}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)} \int_0^{\theta_i} (\theta_i - \mathfrak{q}\tau)^{(\beta-\rho_i-1)} k(\tau)\, d_{\mathfrak{q}}\tau \right], \quad \nu \in [0,1], \\
\mathrm{v}(\nu) = & \frac{\nu^{\beta-1}}{\Delta} \left[-\Lambda_1 \sum_{i=1}^{\mathfrak{c}} \frac{c_i}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)} \int_0^{\zeta_i} (\zeta_i - \mathfrak{q}\tau)^{(\alpha-\eta_i-1)} h(\tau)\, d_{\mathfrak{q}}\tau \right. \\
& + \frac{\Lambda_3}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)} \int_0^1 (1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)} h(\tau)\, d_{\mathfrak{q}}\tau \\
& \left. - \Lambda_3 \sum_{i=1}^{\mathfrak{a}} \frac{a_i}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)} \int_0^{\xi_i} (\xi_i - \mathfrak{q}\tau)^{(\alpha-\varrho_i-1)} h(\tau)\, d_{\mathfrak{q}}\tau \right] \\
& - \frac{1}{\Gamma_{\mathfrak{q}}(\beta)} \int_0^{\nu} (\nu - \mathfrak{q}\tau)^{(\beta-1)} k(\tau)\, d_{\mathfrak{q}}\tau \\
& + \frac{\nu^{\beta-1}}{\Delta} \left[\frac{\Lambda_1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)} \int_0^1 (1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)} k(\tau)\, d_{\mathfrak{q}}\tau \right. \\
& - \Lambda_1 \sum_{i=1}^{\mathfrak{d}} \frac{d_i}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)} \int_0^{\theta_i} (\theta_i - \mathfrak{q}\tau)^{(\beta-\rho_i-1)} k(\tau)\, d_{\mathfrak{q}}\tau \\
& \left. - \Lambda_3 \sum_{i=1}^{\mathfrak{b}} \frac{b_i}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)} \int_0^{\omega_i} (\omega_i - \mathfrak{q}\tau)^{(\beta-\sigma_i-1)} k(\tau)\, d_{\mathfrak{q}}\tau \right], \quad \nu \in [0,1].
\end{aligned} \tag{11}
$$

By the definition of fractional $\mathfrak{q}$-integrals, we obtain the next lemma.

Lemma 2. *The following relations are satisfied:*

(a) $\displaystyle \frac{1}{\Gamma_{\mathfrak{q}}(\alpha)} \int_0^{\nu} (\nu - \mathfrak{q}\tau)^{(\alpha-1)}\, d_{\mathfrak{q}}\tau = \frac{\nu^{\alpha}}{\Gamma_{\mathfrak{q}}(\alpha+1)} \left(= (I_{\mathfrak{q}}^{\alpha}1)(\nu)\right), \quad \nu \geq 0,$

(b) $\displaystyle \frac{1}{\Gamma_{\mathfrak{q}}(\beta)} \int_0^{\nu} (\nu - \mathfrak{q}\tau)^{(\beta-1)}\, d_{\mathfrak{q}}\tau = \frac{\nu^{\beta}}{\Gamma_{\mathfrak{q}}(\beta+1)} \left(= (I_{\mathfrak{q}}^{\beta}1)(\nu)\right), \quad \nu \geq 0,$

(c) $\displaystyle \frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)} \int_0^1 (1 - \mathfrak{q}\tau)^{(\alpha-\varsigma-1)}\, d_{\mathfrak{q}}\tau = \frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma+1)},$

(d) $\displaystyle \frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)} \int_0^1 (1 - \mathfrak{q}\tau)^{(\beta-\vartheta-1)}\, d_{\mathfrak{q}}\tau = \frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta+1)},$

(e) $\displaystyle \frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)} \int_0^{\xi_i} (\xi_i - \mathfrak{q}\tau)^{(\alpha-\varrho_i-1)}\, d_{\mathfrak{q}}\tau = \frac{\xi_i^{\alpha-\varrho_i}}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i+1)}, \quad i = 1, \ldots, \mathfrak{a},$

(f) $\displaystyle \frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)} \int_0^{\zeta_i} (\zeta_i - \mathfrak{q}\tau)^{(\alpha-\eta_i-1)}\, d_{\mathfrak{q}}\tau = \frac{\zeta_i^{\alpha-\eta_i}}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i+1)}, \quad i = 1, \ldots, \mathfrak{c},$

(g) $\displaystyle \frac{1}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)} \int_0^{\omega_i} (\omega_i - \mathfrak{q}\tau)^{(\beta-\sigma_i-1)}\, d_{\mathfrak{q}}\tau = \frac{\omega_i^{\beta-\sigma_i}}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i+1)}, \quad i = 1, \ldots, \mathfrak{b},$

(h) $\displaystyle \frac{1}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)} \int_0^{\theta_i} (\theta_i - \mathfrak{q}\tau)^{(\beta-\rho_i-1)}\, d_{\mathfrak{q}}\tau = \frac{\theta_i^{\beta-\rho_i}}{\Gamma_{\mathfrak{q}}(\beta-\rho_i+1)}, \quad i = 1, \ldots, \mathfrak{d}.$

Lemma 3 ([1]). *If* $\mathrm{w} \in C[0,1]$, *then for* $\kappa > 0$, *we have*

$$|\mathrm{I}_{\mathrm{q}}^{\kappa}\mathrm{w}(\nu)| \leq \frac{\|\mathrm{w}\|}{\Gamma_{\mathrm{q}}(\kappa+1)}, \quad \forall\, \nu \in [0,1], \tag{12}$$

where $\|\mathrm{w}\| = \sup_{\nu\in[0,1]} |\mathrm{w}(\nu)|$.

We consider now the Banach space $\mathcal{U} = C([0,1],\mathbb{R})$ with the supremum norm $\|\mathrm{u}\| = \sup_{\nu\in[0,1]} |\mathrm{u}(\nu)|$, and the Banach space $\mathcal{V} = \mathcal{U}\times\mathcal{U}$ with the norm $\|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}} = \|\mathrm{u}\| + \|\mathrm{v}\|$. We define the operator $\mathcal{E} : \mathcal{V} \to \mathcal{V}$, $\mathcal{E}(\mathrm{u},\mathrm{v}) = (\mathcal{E}_1(\mathrm{u},\mathrm{v}), \mathcal{E}_2(\mathrm{u},\mathrm{v}))$, with $\mathcal{E}_1$, $\mathcal{E}_2 : \mathcal{V} \to \mathcal{U}$ given by

$$\begin{aligned}
\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu) &= -\frac{1}{\Gamma_{\mathrm{q}}(\alpha)}\int_0^{\nu}(\nu-\mathrm{q}\tau)^{(\alpha-1)}\mathcal{F}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&+\frac{\nu^{\alpha-1}}{\Delta}\Bigg[\frac{\Lambda_4}{\Gamma_{\mathrm{q}}(\alpha-\varsigma)}\int_0^1(1-\mathrm{q}\tau)^{(\alpha-\varsigma-1)}\mathcal{F}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&-\Lambda_4\sum_{i=1}^{\mathfrak{a}}\frac{a_i}{\Gamma_{\mathrm{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathrm{q}\tau)^{(\alpha-\varrho_i-1)}\mathcal{F}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&-\Lambda_2\sum_{i=1}^{\mathfrak{c}}\frac{c_i}{\Gamma_{\mathrm{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathrm{q}\tau)^{(\alpha-\eta_i-1)}\mathcal{F}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau\Bigg] \\
&+\frac{\nu^{\alpha-1}}{\Delta}\Bigg[-\Lambda_4\sum_{i=1}^{\mathfrak{b}}\frac{b_i}{\Gamma_{\mathrm{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathrm{q}\tau)^{(\beta-\sigma_i-1)}\mathcal{G}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&+\frac{\Lambda_2}{\Gamma_{\mathrm{q}}(\beta-\vartheta)}\int_0^1(1-\mathrm{q}\tau)^{(\beta-\vartheta-1)}\mathcal{G}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&-\Lambda_2\sum_{i=1}^{\mathfrak{d}}\frac{d_i}{\Gamma_{\mathrm{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathrm{q}\tau)^{(\beta-\rho_i-1)}\mathcal{G}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau\Bigg], \quad \nu\in[0,1], \\
\mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu) &= \frac{\nu^{\beta-1}}{\Delta}\Bigg[-\Lambda_1\sum_{i=1}^{\mathfrak{c}}\frac{c_i}{\Gamma_{\mathrm{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathrm{q}\tau)^{(\alpha-\eta_i-1)}\mathcal{F}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&+\frac{\Lambda_3}{\Gamma_{\mathrm{q}}(\alpha-\varsigma)}\int_0^1(1-\mathrm{q}\tau)^{(\alpha-\varsigma-1)}\mathcal{F}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&-\Lambda_3\sum_{i=1}^{\mathfrak{a}}\frac{a_i}{\Gamma_{\mathrm{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathrm{q}\tau)^{(\alpha-\varrho_i-1)}\mathcal{F}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau\Bigg] \\
&-\frac{1}{\Gamma_{\mathrm{q}}(\beta)}\int_0^{\nu}(\nu-\mathrm{q}\tau)^{(\beta-1)}\mathcal{G}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&+\frac{\nu^{\beta-1}}{\Delta}\Bigg[\frac{\Lambda_1}{\Gamma_{\mathrm{q}}(\beta-\vartheta)}\int_0^1(1-\mathrm{q}\tau)^{(\beta-\vartheta-1)}\mathcal{G}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&-\Lambda_1\sum_{i=1}^{\mathfrak{d}}\frac{d_i}{\Gamma_{\mathrm{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathrm{q}\tau)^{(\beta-\rho_i-1)}\mathcal{G}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau \\
&-\Lambda_3\sum_{i=1}^{\mathfrak{b}}\frac{b_i}{\Gamma_{\mathrm{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathrm{q}\tau)^{(\beta-\sigma_i-1)}\mathcal{G}_{\mathrm{uv}}(\tau)\,d_{\mathrm{q}}\tau\Bigg], \quad \nu\in[0,1],
\end{aligned} \tag{13}$$

for $(\mathrm{u},\mathrm{v}) \in \mathcal{V}$, where $\mathcal{F}_{\mathrm{uv}}(\tau) = \mathcal{F}(\tau,\mathrm{u}(\tau),\mathrm{v}(\tau),\mathrm{I}_{\mathrm{q}}^{\delta_1}\mathrm{u}(\tau),\mathrm{I}_{\mathrm{q}}^{\gamma_1}\mathrm{v}(\tau))$, $\mathcal{G}_{\mathrm{uv}}(\tau) = \mathcal{G}(\tau,\mathrm{u}(\tau),\mathrm{v}(\tau),\mathrm{I}_{\mathrm{q}}^{\delta_2}\mathrm{u}(\tau),\mathrm{I}_{\mathrm{q}}^{\gamma_2}\mathrm{v}(\tau))$, for any $\tau \in [0,1]$.

By Lemma 1, we see that (u,v) is a solution of problem (1), (2) if and only if (u,v) is a fixed point of operator $\mathcal{E}$.

3. Existence of Solutions

In this section, we will outline the principal existence results for the problem defined by Equations (1) and (2).

We introduce the fundamental assumptions that form the basis of our theorems.

$(J1)$ $\mathfrak{q} \in (0,1)$, α, $\beta \in \mathbb{R}$, $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n,m \in \mathbb{N}$, $n,m \geq 2$; $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d} \in \mathbb{N}$; $\varsigma, \varrho_i, \eta_k \in [0,\alpha-1)$; $\vartheta, \sigma_j, \rho_\iota \in [0,\beta-1)$; $a_i, b_j, c_k, d_\iota \in \mathbb{R}$; $\xi_i, \omega_j, \zeta_k, \theta_\iota \in (0,1)$, for all $i = 1,\ldots,\mathfrak{a}$, $j = 1,\ldots,\mathfrak{b}$, $k = 1,\ldots,\mathfrak{c}$, $\iota = 1,\ldots,\mathfrak{d}$; $\delta_\kappa, \gamma_\kappa > 0$ for $\kappa = 1,2$; $\Delta \neq 0$ (given by (10)).

We also define the constants

$$\begin{aligned}
\Upsilon_1 &= \frac{1}{\Gamma_{\mathfrak{q}}(\alpha+1)} + \frac{1}{|\Delta|}\left[|\Lambda_4|\frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma+1)} + |\Lambda_4|\sum_{i=1}^{\mathfrak{a}}|a_i|\frac{\xi_i^{\alpha-\varrho_i}}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i+1)}\right. \\
&\quad \left. +|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}|c_i|\frac{\zeta_i^{\alpha-\eta_i}}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i+1)}\right], \\
\Upsilon_2 &= \frac{1}{|\Delta|}\left[|\Lambda_2|\frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta+1)} + |\Lambda_4|\sum_{i=1}^{\mathfrak{b}}|b_i|\frac{\omega_i^{\beta-\sigma_i}}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i+1)}\right. \\
&\quad \left. +|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}|d_i|\frac{\theta_i^{\beta-\rho_i}}{\Gamma_{\mathfrak{q}}(\beta-\rho_i+1)}\right] \\
\Upsilon_3 &= \frac{1}{|\Delta|}\left[|\Lambda_3|\frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma+1)} + |\Lambda_3|\sum_{i=1}^{\mathfrak{a}}|a_i|\frac{\xi_i^{\alpha-\varrho_i}}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i+1)}\right. \\
&\quad \left. +|\Lambda_1|\sum_{i=1}^{\mathfrak{c}}|c_i|\frac{\zeta_i^{\alpha-\eta_i}}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i+1)}\right], \\
\Upsilon_4 &= \frac{1}{\Gamma_{\mathfrak{q}}(\beta+1)} + \frac{1}{|\Delta|}\left[|\Lambda_1|\frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta+1)} + |\Lambda_3|\sum_{i=1}^{\mathfrak{b}}|b_i|\frac{\omega_i^{\beta-\sigma_i}}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i+1)}\right. \\
&\quad \left. +|\Lambda_1|\sum_{i=1}^{\mathfrak{d}}|d_i|\frac{\theta_i^{\beta-\rho_i}}{\Gamma_{\mathfrak{q}}(\beta-\rho_i+1)}\right].
\end{aligned} \tag{14}$$

Under assumption $(J1)$, we remark that $\Upsilon_1 > 0$, $\Upsilon_2 \geq 0$, $\Upsilon_3 \geq 0$, $\Upsilon_4 > 0$, and so $\Upsilon_1 + \Upsilon_3 > 0$, $\Upsilon_2 + \Upsilon_4 > 0$.

The initial existence and uniqueness theorem for problem (1), (2) is as follows, relying on the Banach contraction mapping principle, as detailed in [32].

Theorem 1. *Suppose that $(J1)$ holds. In addition, we assume that the functions $\mathcal{F}$, $\mathcal{G} : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous and satisfy the condition*

$(J2)$ *There exist the functions $\mathcal{H}_i$, $\mathcal{K}_i \in C([0,1],\mathbb{R}_+)$, $i = 1,\ldots,4$, $(\mathbb{R}_+ = [0,\infty))$, such that*

$$\begin{aligned}
|\mathcal{F}(\nu,\mathrm{u}_1,\mathrm{u}_2,\mathrm{u}_3,\mathrm{u}_4) - \mathcal{F}(\nu,\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4)| &\leq \sum_{i=1}^{4}\mathcal{H}_i(\nu)|\mathrm{u}_i - \mathrm{v}_i|, \\
|\mathcal{G}(\nu,\mathrm{u}_1,\mathrm{u}_2,\mathrm{u}_3,\mathrm{u}_4) - \mathcal{G}(\nu,\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4)| &\leq \sum_{i=1}^{4}\mathcal{K}_i(\nu)|\mathrm{u}_i - \mathrm{v}_i|,
\end{aligned} \tag{15}$$

for all $\nu \in [0,1]$ and $\mathrm{u}_i, \mathrm{v}_i \in \mathbb{R}$, $i = 1,\ldots,4$.

If

$$\Theta_0 < 1, \tag{16}$$

where $\Theta_0 = \max\{\Theta_1,\Theta_2\}$,

$$\begin{aligned}
\Theta_1 &= \left(\mathfrak{h}_1^* + \frac{\mathfrak{h}_3^*}{\Gamma_{\mathfrak{q}}(\delta_1+1)}\right)(\Upsilon_1+\Upsilon_3) + \left(\mathfrak{k}_1^* + \frac{\mathfrak{k}_3^*}{\Gamma_{\mathfrak{q}}(\delta_2+1)}\right)(\Upsilon_2+\Upsilon_4), \\
\Theta_2 &= \left(\mathfrak{h}_2^* + \frac{\mathfrak{h}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_1+1)}\right)(\Upsilon_1+\Upsilon_3) + \left(\mathfrak{k}_2^* + \frac{\mathfrak{k}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_2+1)}\right)(\Upsilon_2+\Upsilon_4),
\end{aligned} \tag{17}$$

and $\mathfrak{h}_i^* = \sup_{\nu\in[0,1]} \mathcal{H}_i(\nu)$, $\mathfrak{k}_i^* = \sup_{\nu\in[0,1]} \mathcal{K}_i(\nu)$, $i = 1,\ldots,4$, *then the boundary value problem (1), (2) has a unique solution* $(\mathrm{u}(\nu),\mathrm{v}(\nu))$, $\nu\in[0,1]$.

Proof. We denote by $\Xi_1 = \sup_{\nu\in[0,1]} |\mathcal{F}(\nu,0,0,0,0)|$ and $\Xi_2 = \sup_{\nu\in[0,1]} |\mathcal{G}(\nu,0,0,0,0)|$. We consider the positive number

$$R \ge \frac{\Xi_1(\Upsilon_1+\Upsilon_3)+\Xi_2(\Upsilon_2+\Upsilon_4)}{1-\Theta_0}, \tag{18}$$

and let the set $\Omega = \{(\mathrm{u},\mathrm{v})\in\mathcal{V},\ \|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}} \le R\}$.

We will show firstly that $\mathcal{E}(\Omega)\subset\Omega$. For this, let $(\mathrm{u},\mathrm{v})\in\Omega$, that is $\|\mathrm{u}\|+\|\mathrm{v}\|\le R$. Then, by $(J1)$ and Lemma 3, we obtain for all $\nu\in[0,1]$

$$\begin{aligned}
&|\mathcal{F}_{\mathrm{uv}}(\nu)| = |\mathcal{F}(\nu,\mathrm{u}(\nu),\mathrm{v}(\nu),\mathrm{I}_{\mathrm{q}}^{\delta_1}\mathrm{u}(\nu),\mathrm{I}_{\mathrm{q}}^{\gamma_1}\mathrm{v}(\nu))|\\
&\quad\le |\mathcal{F}(\nu,\mathrm{u}(\nu),\mathrm{v}(\nu),\mathrm{I}_{\mathrm{q}}^{\delta_1}\mathrm{u}(\nu),\mathrm{I}_{\mathrm{q}}^{\gamma_1}\mathrm{v}(\nu)) - \mathcal{F}(\nu,0,0,0,0)| + |\mathcal{F}(\nu,0,0,0,0)|\\
&\quad\le \mathcal{H}_1(\nu)|\mathrm{u}(\nu)| + \mathcal{H}_2(\nu)|\mathrm{v}(\nu)| + \mathcal{H}_3(\nu)|\mathrm{I}_{\mathrm{q}}^{\delta_1}\mathrm{u}(\nu)| + \mathcal{H}_4(\nu)|\mathrm{I}_{\mathrm{q}}^{\gamma_1}\mathrm{v}(\nu)| + \Xi_1\\
&\quad\le \mathfrak{h}_1^*\|\mathrm{u}\| + \mathfrak{h}_2^*\|\mathrm{v}\| + \mathfrak{h}_3^*\frac{\|\mathrm{u}\|}{\Gamma_{\mathrm{q}}(\delta_1+1)} + \mathfrak{h}_4^*\frac{\|\mathrm{v}\|}{\Gamma_{\mathrm{q}}(\gamma_1+1)} + \Xi_1\\
&\quad= \left(\mathfrak{h}_1^* + \frac{\mathfrak{h}_3^*}{\Gamma_{\mathrm{q}}(\delta_1+1)}\right)\|\mathrm{u}\| + \left(\mathfrak{h}_2^* + \frac{\mathfrak{h}_4^*}{\Gamma_{\mathrm{q}}(\gamma_1+1)}\right)\|\mathrm{v}\| + \Xi_1 =: A_{\mathrm{uv}},\\
&|\mathcal{G}_{\mathrm{uv}}(\nu)| = |\mathcal{G}(\nu,\mathrm{u}(\nu),\mathrm{v}(\nu),\mathrm{I}_{\mathrm{q}}^{\delta_2}\mathrm{u}(\nu),\mathrm{I}_{\mathrm{q}}^{\gamma_2}\mathrm{v}(\nu))|\\
&\quad\le |\mathcal{G}(\nu,\mathrm{u}(\nu),\mathrm{v}(\nu),\mathrm{I}_{\mathrm{q}}^{\delta_2}\mathrm{u}(\nu),\mathrm{I}_{\mathrm{q}}^{\gamma_2}\mathrm{v}(\nu)) - \mathcal{G}(\nu,0,0,0,0)| + |\mathcal{G}(\nu,0,0,0,0)|\\
&\quad\le \mathcal{K}_1(\nu)|\mathrm{u}(\nu)| + \mathcal{K}_2(\nu)|\mathrm{v}(\nu)| + \mathcal{K}_3(\nu)|\mathrm{I}_{\mathrm{q}}^{\delta_2}\mathrm{u}(\nu)| + \mathcal{K}_4(\nu)|\mathrm{I}_{\mathrm{q}}^{\gamma_2}\mathrm{v}(\nu)| + \Xi_2\\
&\quad\le \mathfrak{k}_1^*\|\mathrm{u}\| + \mathfrak{k}_2^*\|\mathrm{v}\| + \mathfrak{k}_3^*\frac{\|\mathrm{u}\|}{\Gamma_{\mathrm{q}}(\delta_2+1)} + \mathfrak{k}_4^*\frac{\|\mathrm{v}\|}{\Gamma_{\mathrm{q}}(\gamma_2+1)} + \Xi_2\\
&\quad= \left(\mathfrak{k}_1^* + \frac{\mathfrak{k}_3^*}{\Gamma_q(\delta_2+1)}\right)\|\mathrm{u}\| + \left(\mathfrak{k}_2^* + \frac{\mathfrak{k}_4^*}{\Gamma_q(\gamma_2+1)}\right)\|\mathrm{v}\| + \Xi_2 =: B_{\mathrm{uv}}.
\end{aligned} \tag{19}$$

Therefore, we find

$$\begin{aligned}
&|\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu)| \le \frac{1}{\Gamma_{\mathrm{q}}(\alpha)}\int_0^{\nu}(\nu-\mathrm{q}\tau)^{(\alpha-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau\\
&\quad+\frac{1}{|\Delta|}\Bigg[|\Lambda_4|\frac{1}{\Gamma_q(\alpha-\varsigma)}\int_0^1(1-\mathrm{q}\tau)^{(\alpha-\varsigma-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_q\tau\\
&\quad+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_{\mathrm{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathrm{q}\tau)^{(\alpha-\varrho_i-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau\\
&\quad+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_q(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathrm{q}\tau)^{(\alpha-\eta_i-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau\Bigg]\\
&\quad+\frac{1}{|\Delta|}\Bigg[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_q(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathrm{q}\tau)^{(\beta-\sigma_i-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_q\tau\\
&\quad+|\Lambda_2|\frac{1}{\Gamma_{\mathrm{q}}(\beta-\vartheta)}\int_0^1(1-\mathrm{q}\tau)^{(\beta-\vartheta-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau\\
&\quad+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_{\mathrm{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathrm{q}\tau)^{(\beta-\rho_i-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau\Bigg]\\
&\le A_{\mathrm{uv}}\Bigg\{\frac{1}{\Gamma_{\mathrm{q}}(\alpha)}\int_0^{\nu}(\nu-\mathrm{q}\tau)^{(\alpha-1)}\,d_{\mathrm{q}}\tau + \frac{1}{|\Delta|}\Bigg[|\Lambda_4|\frac{1}{\Gamma_{\mathrm{q}}(\alpha-\varsigma)}\int_0^1(1-\mathrm{q}\tau)^{(\alpha-\varsigma-1)}\,d_{\mathrm{q}}\tau
\end{aligned}$$

$$
\begin{aligned}
&+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-q\tau)^{(\alpha-\varrho_i-1)}\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathfrak{q}\tau)^{(\alpha-\eta_i-1)}\,d_{\mathfrak{q}}\tau\Bigg]\Bigg\}\\
&+B_{\mathrm{uv}}\frac{1}{|\Delta|}\Bigg[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathfrak{q}\tau)^{(\beta-\sigma_i-1)}\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_2|\frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)}\int_0^1(1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)}\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathfrak{q}\tau)^{(\beta-\rho_i-1)}\,d_{\mathfrak{q}}\tau\Bigg]\\
=&\,A_{\mathrm{uv}}\Bigg\{\frac{\nu^{\alpha}}{\Gamma_{\mathfrak{q}}(\alpha+1)}+\frac{1}{|\Delta|}\Bigg[|\Lambda_4|\frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma+1)}+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}|a_i|\frac{\xi_i^{\alpha-\varrho_i}}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i+1)}\\
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}|c_i|\frac{\zeta_i^{\alpha-\eta_i}}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i+1)}\Bigg]\Bigg\}\\
&+B_{\mathrm{uv}}\frac{1}{|\Delta|}\Bigg[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}|b_i|\frac{\omega_i^{\beta-\sigma_i}}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i+1)}+|\Lambda_2|\frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta+1)}\\
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}|d_i|\frac{\theta_i^{\beta-\rho_i}}{\Gamma_{\mathfrak{q}}(\beta-\rho_i+1)}\Bigg]\le A_{\mathrm{uv}}\Upsilon_1+B_{\mathrm{uv}}\Upsilon_2,\ \ \forall\,\nu\in[0,1],
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
|\mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu)|\le&\,\frac{1}{|\Delta|}\Bigg[|\Lambda_1|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathfrak{q}\tau)^{(\alpha-\eta_i-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_3|\frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)}\int_0^1(1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_3|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathfrak{q}\tau)^{(\alpha-\varrho_i-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\Bigg]\\
&+\frac{1}{\Gamma_{\mathfrak{q}}(\beta)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\beta-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\\
&+\frac{1}{|\Delta|}\Bigg[|\Lambda_1|\frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)}\int_0^1(1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_1|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathfrak{q}\tau)^{(\beta-\rho_i-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_3|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathfrak{q}\tau)^{(\beta-\sigma_i-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\Bigg]\\
\le&\,A_{\mathrm{uv}}\frac{1}{|\Delta|}\Bigg[|\Lambda_1|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathfrak{q}\tau)^{(\alpha-\eta_i-1)}\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_3|\frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)}\int_0^1(1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)}\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_3|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathfrak{q}\tau)^{(\alpha-\varrho_i-1)}\,d_{\mathfrak{q}}\tau\Bigg]\\
&+B_{\mathrm{uv}}\Bigg\{\frac{1}{\Gamma_{\mathfrak{q}}(\beta)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\beta-1)}\,d_{\mathfrak{q}}\tau+\frac{1}{|\Delta|}\Bigg[|\Lambda_1|\frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)}\int_0^1(1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)}\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_1|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathfrak{q}\tau)^{(\beta-\rho_i-1)}\,d_{\mathfrak{q}}\tau\\
&+|\Lambda_3|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathfrak{q}\tau)^{(\beta-\sigma_i-1)}\,d_{\mathfrak{q}}\tau\Bigg]\Bigg\}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&= A_{\mathrm{uv}}\frac{1}{|\Delta|}\Bigg[|\Lambda_1|\sum_{i=1}^{\mathfrak{c}}|c_i|\frac{\zeta_i^{\alpha-\eta_i}}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i+1)}+|\Lambda_3|\frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma+1)}\\
&\quad+|\Lambda_3|\sum_{i=1}^{\mathfrak{a}}|a_i|\frac{\xi_i^{\alpha-\varrho_i}}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i+1)}\Bigg]\\
&\quad+B_{\mathrm{uv}}\Bigg\{\frac{\nu^{\beta}}{\Gamma_{\mathfrak{q}}(\beta+1)}+\frac{1}{|\Delta|}\Bigg[|\Lambda_1|\frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta+1)}+|\Lambda_1|\sum_{i=1}^{\mathfrak{d}}|d_i|\frac{\theta_i^{\beta-\rho_i}}{\Gamma_{\mathfrak{q}}(\beta-\rho_i+1)}\\
&\quad+|\Lambda_3|\sum_{i=1}^{\mathfrak{b}}|b_i|\frac{\omega_i^{\beta-\sigma_i}}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i+1)}\Bigg]\Bigg\}\le A_{\mathrm{uv}}\Upsilon_3+B_{\mathrm{uv}}\Upsilon_4,\ \ \forall\,\nu\in[0,1].
\end{aligned}
$$

Therefore, by (20), (21) and (18), we deduce

$$
\begin{aligned}
&\|\mathcal{E}(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}=\|\mathcal{E}_1(\mathrm{u},\mathrm{v})\|+\|\mathcal{E}_2(\mathrm{u},\mathrm{v})\|\le A_{\mathrm{uv}}(\Upsilon_1+\Upsilon_3)+B_{\mathrm{uv}}(\Upsilon_2+\Upsilon_4)\\
&=\Bigg[\Bigg(\mathfrak{h}_1^*+\frac{\mathfrak{h}_3^*}{\Gamma_{\mathfrak{q}}(\delta_1+1)}\Bigg)\|\mathrm{u}\|+\Bigg(\mathfrak{h}_2^*+\frac{\mathfrak{h}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_1+1)}\Bigg)\|\mathrm{v}\|+\Xi_1\Bigg](\Upsilon_1+\Upsilon_3)\\
&\quad+\Bigg[\Bigg(\mathfrak{k}_1^*+\frac{\mathfrak{k}_3^*}{\Gamma_{\mathfrak{q}}(\delta_2+1)}\Bigg)\|\mathrm{u}\|+\Bigg(\mathfrak{k}_2^*+\frac{\mathfrak{k}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_2+1)}\Bigg)\|\mathrm{v}\|+\Xi_2\Bigg](\Upsilon_2+\Upsilon_4)\\
&=\Bigg[\Bigg(\mathfrak{h}_1^*+\frac{\mathfrak{h}_3^*}{\Gamma_{\mathfrak{q}}(\delta_1+1)}\Bigg)(\Upsilon_1+\Upsilon_3)+\Bigg(\mathfrak{k}_1^*+\frac{\mathfrak{k}_3^*}{\Gamma_{\mathfrak{q}}(\delta_2+1)}\Bigg)(\Upsilon_2+\Upsilon_4)\Bigg]\|\mathrm{u}\|\\
&\quad+\Bigg[\Bigg(\mathfrak{h}_2^*+\frac{\mathfrak{h}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_1+1)}\Bigg)(\Upsilon_1+\Upsilon_3)+\Bigg(\mathfrak{k}_2^*+\frac{\mathfrak{k}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_2+1)}\Bigg)(\Upsilon_2+\Upsilon_4)\Bigg]\|\mathrm{v}\|\\
&\quad+\Xi_1(\Upsilon_1+\Upsilon_3)+\Xi_2(\Upsilon_2+\Upsilon_4)\\
&=\Theta_1\|\mathrm{u}\|+\Theta_2\|\mathrm{v}\|+\Xi_1(\Upsilon_1+\Upsilon_3)+\Xi_2(\Upsilon_2+\Upsilon_4)\\
&\le\Theta_0\|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}+\Xi_1(\Upsilon_1+\Upsilon_3)+\Xi_2(\Upsilon_2+\Upsilon_4)\\
&\le\Theta_0R+\Xi_1(\Upsilon_1+\Upsilon_3)+\Xi_2(\Upsilon_2+\Upsilon_4)\le R.
\end{aligned}\tag{22}
$$

Therefore, we conclude that $\mathcal{E}(\Omega)\subset\Omega$.

Subsequently, we will prove that $\mathcal{E}$ is a contraction. For this, let $(\mathrm{u}_1,\mathrm{v}_1)$, $(\mathrm{u}_2,\mathrm{v}_2)\in\mathcal{V}$. By relations (15), we find for any $\tau\in[0,1]$

$$
\begin{aligned}
&|\mathcal{F}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\le\mathcal{H}_1(\tau)|\mathrm{u}_1(\tau)-\mathrm{u}_2(\tau)|+\mathcal{H}_2(\tau)|\mathrm{v}_1(\tau)-\mathrm{v}_2(\tau)|\\
&\quad+\mathcal{H}_3(\tau)|\mathrm{I}_{\mathfrak{q}}^{\delta_1}\mathrm{u}_1(\tau)-\mathrm{I}_{\mathfrak{q}}^{\delta_1}\mathrm{u}_2(\tau)|+\mathcal{H}_4(\tau)|\mathrm{I}_{\mathfrak{q}}^{\gamma_1}\mathrm{v}_1(\tau)-\mathrm{I}_{\mathfrak{q}}^{\gamma_1}\mathrm{v}_2(\tau)|\\
&\le\mathfrak{h}_1^*\|\mathrm{u}_1-\mathrm{u}_2\|+\mathfrak{h}_2^*\|\mathrm{v}_1-\mathrm{v}_2\|+\frac{\mathfrak{h}_3^*}{\Gamma_{\mathfrak{q}}(\delta_1+1)}\|\mathrm{u}_1-\mathrm{u}_2\|+\frac{\mathfrak{h}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_1+1)}\|\mathrm{v}_1-\mathrm{v}_2\|\\
&=\Bigg(\mathfrak{h}_1^*+\frac{\mathfrak{h}_3^*}{\Gamma_{\mathfrak{q}}(\delta_1+1)}\Bigg)\|\mathrm{u}_1-\mathrm{u}_2\|+\Bigg(\mathfrak{h}_2^*+\frac{\mathfrak{h}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_1+1)}\Bigg)\|\mathrm{v}_1-\mathrm{v}_2\|=:C_{\mathrm{uv}},\\
&|\mathcal{G}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{G}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\le\mathcal{K}_1(\tau)|\mathrm{u}_1(\tau)-\mathrm{u}_2(\tau)|+\mathcal{K}_2(\tau)|\mathrm{v}_1(\tau)-\mathrm{v}_2(\tau)|\\
&\quad+\mathcal{K}_3(\tau)|\mathrm{I}_{\mathfrak{q}}^{\delta_2}\mathrm{u}_1(\tau)-\mathrm{I}_{\mathfrak{q}}^{\delta_2}\mathrm{u}_2(\tau)|+\mathcal{K}_4(\tau)|\mathrm{I}_{\mathfrak{q}}^{\gamma_2}\mathrm{v}_1(\tau)-\mathrm{I}_{\mathfrak{q}}^{\gamma_2}\mathrm{v}_2(\tau)|\\
&\le\mathfrak{k}_1^*\|\mathrm{u}_1-\mathrm{u}_2\|+\mathfrak{k}_2^*\|\mathrm{v}_1-\mathrm{v}_2\|+\frac{\mathfrak{k}_3^*}{\Gamma_{\mathfrak{q}}(\delta_2+1)}\|\mathrm{u}_1-\mathrm{u}_2\|+\frac{\mathfrak{k}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_2+1)}\|\mathrm{v}_1-\mathrm{v}_2\|\\
&=\Bigg(\mathfrak{k}_1^*+\frac{\mathfrak{k}_3^*}{\Gamma_{\mathfrak{q}}(\delta_2+1)}\Bigg)\|\mathrm{u}_1-\mathrm{u}_2\|+\Bigg(\mathfrak{k}_2^*+\frac{\mathfrak{k}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_2+1)}\Bigg)\|\mathrm{v}_1-\mathrm{v}_2\|=:D_{\mathrm{uv}}.
\end{aligned}\tag{23}
$$

Then, for any $\nu\in[0,1]$, we obtain

$$
\begin{aligned}
&|\mathcal{E}_1(\mathrm{u}_1,\mathrm{v}_1)(\nu)-\mathcal{E}_1(\mathrm{u}_2,\mathrm{v}_2)(\nu)|\\
&\le\frac{1}{\Gamma_{\mathfrak{q}}(\alpha)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\alpha-1)}|\mathcal{F}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\,d_{\mathfrak{q}}\tau\\
&\quad+\frac{1}{|\Delta|}\Bigg[\frac{|\Lambda_4|}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)}\int_0^1(1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)}|\mathcal{F}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\,d_{\mathfrak{q}}\tau\\
&\quad+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathfrak{q}\tau)^{\alpha-\varrho_i-1)}|\mathcal{F}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\,d_{\mathfrak{q}}\tau\\
&\quad+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathfrak{q}\tau)^{(\alpha-\eta_i-1)}|\mathcal{F}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\,d_{\mathfrak{q}}\tau\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\Delta|}\Bigg[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_{\mathrm{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathrm{q}\tau)^{(\beta-\sigma_i-1)}|\mathcal{G}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{G}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\,d_{\mathrm{q}}\tau \\
&+\frac{|\Lambda_2|}{\Gamma_{\mathrm{q}}(\beta-\vartheta)}\int_0^{1}(1-\mathrm{q}\tau)^{(\beta-\vartheta-1)}|\mathcal{G}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{G}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\,d_{\mathrm{q}}\tau \\
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_{\mathrm{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathrm{q}\tau)^{(\beta-\rho_i-1)}|\mathcal{G}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{G}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\,d_{\mathrm{q}}\tau\Bigg] \\
\le\ & C_{\mathrm{uv}}\Bigg\{\frac{1}{\Gamma_{\mathrm{q}}(\alpha)}\int_0^{\nu}(\nu-\mathrm{q}\tau)^{(\alpha-1)}\,d_{\mathrm{q}}\tau+\frac{1}{|\Delta|}\Bigg[\frac{|\Lambda_4|}{\Gamma_{\mathrm{q}}(\alpha-\varsigma)}\int_0^{1}(1-\mathrm{q}\tau)^{(\alpha-\varsigma-1)}\,d_{\mathrm{q}}\tau \\
&+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_{\mathrm{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathrm{q}\tau)^{(\alpha-\varrho_i-1)}\,d_{\mathrm{q}}\tau \\
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_{\mathrm{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathrm{q}\tau)^{(\alpha-\eta_i-1)}\,d_{\mathrm{q}}\tau\Bigg]\Bigg\} \\
&+D_{\mathrm{uv}}\frac{1}{|\Delta|}\Bigg[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_{\mathrm{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathrm{q}\tau)^{(\beta-\sigma_i-1)}\,d_{\mathrm{q}}\tau \\
&+\frac{|\Lambda_2|}{\Gamma_{\mathrm{q}}(\beta-\vartheta)}\int_0^{1}(1-\mathrm{q}\tau)^{(\beta-\vartheta-1)}\,d_{\mathrm{q}}\tau \\
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_{\mathrm{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathrm{q}\tau)^{(\beta-\rho_i-1)}\,d_{\mathrm{q}}\tau\Bigg]\le C_{\mathrm{uv}}\Upsilon_1+D_{\mathrm{uv}}\Upsilon_2.
\end{aligned}
\tag{24}
$$

In a similar manner, for any $\nu\in[0,1]$, we deduce

$$
|\mathcal{E}_2(\mathrm{u}_1,\mathrm{v}_1)(\nu)-\mathcal{E}_2(\mathrm{u}_2,\mathrm{v}_2)(\nu)|\le C_{\mathrm{uv}}\Upsilon_3+D_{\mathrm{uv}}\Upsilon_4. \tag{25}
$$

Then, by (24), (25) and (17), we conclude that

$$
\begin{aligned}
&\|\mathcal{E}(\mathrm{u}_1,\mathrm{v}_1)-\mathcal{E}(\mathrm{u}_2,\mathrm{v}_2)\|_{\mathcal{V}}=\|\mathcal{E}_1(\mathrm{u}_1,\mathrm{v}_1)-\mathcal{E}_1(\mathrm{u}_2,\mathrm{v}_2)\|+\|\mathcal{E}_2(\mathrm{u}_1,\mathrm{v}_1)-\mathcal{E}_2(\mathrm{u}_2,\mathrm{v}_2)\| \\
&\le C_{\mathrm{uv}}(\Upsilon_1+\Upsilon_3)+D_{\mathrm{uv}}(\Upsilon_2+\Upsilon_4) \\
&=\left(\mathfrak{h}_1^*+\frac{\mathfrak{h}_3^*}{\Gamma_{\mathrm{q}}(\delta_1+1)}\right)(\Upsilon_1+\Upsilon_3)\|\mathrm{u}_1-\mathrm{u}_2\|+\left(\mathfrak{h}_2^*+\frac{\mathfrak{h}_4^*}{\Gamma_{\mathrm{q}}(\gamma_1+1)}\right)(\Upsilon_1+\Upsilon_3)\|\mathrm{v}_1-\mathrm{v}_2\| \\
&\quad+\left(\mathfrak{k}_1^*+\frac{\mathfrak{k}_3^*}{\Gamma_{\mathrm{q}}(\delta_2+1)}\right)(\Upsilon_2+\Upsilon_4)\|\mathrm{u}_1-\mathrm{u}_2\|+\left(\mathfrak{k}_2^*+\frac{\mathfrak{k}_4^*}{\Gamma_{\mathrm{q}}(\gamma_2+1)}\right)(\Upsilon_2+\Upsilon_4)\|\mathrm{v}_1-\mathrm{v}_2\| \\
&=\left[\left(\mathfrak{h}_1^*+\frac{\mathfrak{h}_3^*}{\Gamma_{\mathrm{q}}(\delta_1+1)}\right)(\Upsilon_1+\Upsilon_3)+\left(\mathfrak{k}_1^*+\frac{\mathfrak{k}_3^*}{\Gamma_{\mathrm{q}}(\delta_2+1)}\right)(\Upsilon_2+\Upsilon_4)\right]\|\mathrm{u}_1-\mathrm{u}_2\| \\
&\quad+\left[\left(\mathfrak{h}_2^*+\frac{\mathfrak{h}_4^*}{\Gamma_{\mathrm{q}}(\gamma_1+1)}\right)(\Upsilon_1+\Upsilon_3)+\left(\mathfrak{k}_2^*+\frac{\mathfrak{k}_4^*}{\Gamma_{\mathrm{q}}(\gamma_2+1)}\right)(\Upsilon_2+\Upsilon_4)\right]\|\mathrm{v}_1-\mathrm{v}_2\| \\
&=\Theta_1\|\mathrm{u}_1-\mathrm{u}_2\|+\Theta_2\|\mathrm{v}_1-\mathrm{v}_2\|\le\Theta_0(\|\mathrm{u}_1-\mathrm{u}_2\|+\|\mathrm{v}_1-\mathrm{v}_2\|) \\
&=\Theta_0\|(\mathrm{u}_1,\mathrm{v}_1)-(\mathrm{u}_2,\mathrm{v}_2)\|_{\mathcal{V}}.
\end{aligned}
\tag{26}
$$

By (16) and (26), we deduce that $\mathcal{E}$ is a contraction operator. Therefore, by the Banach contraction mapping principle, the operator $\mathcal{E}$ has a unique fixed point $(\mathrm{u}^*,\mathrm{v}^*)\in\Omega$. Therefore, problem (1), (2) has a unique solution $(\mathrm{u}^*(\nu),\mathrm{v}^*(\nu))$, $\nu\in[0,1]$ with $\|\mathrm{u}^*\|+\|\mathrm{v}^*\|\le R$. Moreover, for any $(\mathrm{u}_0,\mathrm{v}_0)\in\Omega$, the sequence $((\mathrm{u}_n,\mathrm{v}_n))_{n\ge0}$ defined by $(\mathrm{u}_n,\mathrm{v}_n)=\mathcal{E}(\mathrm{u}_{n-1},\mathrm{v}_{n-1})$ for $n\ge1$ converges to $(\mathrm{u}^*,\mathrm{v}^*)$ as $n\to\infty$. By the proof of Banach theorem, we obtain the error estimate

$$
\|(\mathrm{u}_n,\mathrm{v}_n)-(\mathrm{u}^*,\mathrm{v}^*)\|_{\mathcal{V}}\le\frac{\Theta_0^n}{1-\Theta_0}\|(\mathrm{u}_1,\mathrm{v}_1)-(\mathrm{u}_0,\mathrm{v}_0)\|_{\mathcal{V}}. \tag{27}
$$

□

Corollary 1. *Suppose that* $(J1)$ *holds. In addition, we assume that the functions* $\mathcal{F},\mathcal{G}:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ *are continuous and*

$(J2)'$ *There exist* $L_i \geq 0$, $M_i \geq 0$, $i = 1, \ldots, 4$ *such that*

$$\begin{aligned} |\mathcal{F}(\nu, \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4) - \mathcal{F}(\nu, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4)| &\leq \sum_{i=1}^{4} L_i |\mathrm{u}_i - \mathrm{v}_i|, \\ |\mathcal{G}(\nu, \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4) - \mathcal{G}(\nu, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4)| &\leq \sum_{i=1}^{4} M_i |\mathrm{u}_i - \mathrm{v}_i|, \end{aligned} \tag{28}$$

for all $\nu \in [0,1]$ *and* $\mathrm{u}_i, \mathrm{v}_i \in \mathbb{R}$, $i = 1, \ldots, 4$.

If $\Theta_0 < 1$, *where* $\Theta_0 = \max\{\Theta_1, \Theta_2\}$,

$$\begin{aligned} \Theta_1 &= \left(L_1 + \frac{L_3}{\Gamma_{\mathrm{q}}(\delta_1 + 1)} \right)(\Upsilon_1 + \Upsilon_3) + \left(M_1 + \frac{M_3}{\Gamma_{\mathrm{q}}(\delta_2 + 1)} \right)(\Upsilon_2 + \Upsilon_4), \\ \Theta_2 &= \left(L_2 + \frac{L_4}{\Gamma_{\mathrm{q}}(\gamma_1 + 1)} \right)(\Upsilon_1 + \Upsilon_3) + \left(M_2 + \frac{M_4}{\Gamma_{\mathrm{q}}(\gamma_2 + 1)} \right)(\Upsilon_2 + \Upsilon_4), \end{aligned} \tag{29}$$

then the boundary value problem (1), (2) has a unique solution $(\mathrm{u}(\nu), \mathrm{v}(\nu))$, $\nu \in [0,1]$.

The following two outcomes regarding the existence of solutions to problem (1), (2) rely on the Krasnosel'skii fixed point theorem applied to the combination of two operators (refer to [33] for details).

Theorem 2. *Suppose that assumptions* $(J1)$ *and* $(J2)$ *hold. In addition, we assume that the functions* $\mathcal{F}, \mathcal{G} : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *are continuous and*

$(J3)$ *There exist the functions* Φ, $\Psi \in C([0,1], \mathbb{R}_+)$ *such that*

$$|\mathcal{F}(\nu, \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4)| \leq \Phi(\nu), \quad |\mathcal{G}(\nu, \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4)| \leq \Psi(\nu), \tag{30}$$

for all $\nu \in [0,1]$, $\mathrm{u}_i \in \mathbb{R}$, $i = 1, \ldots, 4$.

If

$$\mathfrak{L}_0 < 1, \tag{31}$$

where $\mathfrak{L}_0 = \max\{\mathfrak{L}_1, \mathfrak{L}_2\}$ *with*

$$\begin{aligned} \mathfrak{L}_1 &= \left(\mathfrak{h}_1^* + \frac{\mathfrak{h}_3^*}{\Gamma_{\mathrm{q}}(\delta_1 + 1)} \right) \left(\Upsilon_1 + \Upsilon_3 - \frac{1}{\Gamma_{\mathrm{q}}(\alpha + 1)} \right) \\ &\quad + \left(\mathfrak{k}_1^* + \frac{\mathfrak{k}_3^*}{\Gamma_{\mathrm{q}}(\delta_2 + 1)} \right) \left(\Upsilon_2 + \Upsilon_4 - \frac{1}{\Gamma_{\mathrm{q}}(\beta + 1)} \right), \\ \mathfrak{L}_2 &= \left(\mathfrak{h}_2^* + \frac{\mathfrak{h}_4^*}{\Gamma_{\mathrm{q}}(\gamma_1 + 1)} \right) \left(\Upsilon_1 + \Upsilon_3 - \frac{1}{\Gamma_{\mathrm{q}}(\alpha + 1)} \right) \\ &\quad + \left(\mathfrak{k}_2^* + \frac{\mathfrak{k}_4^*}{\Gamma_{\mathrm{q}}(\gamma_2 + 1)} \right) \left(\Upsilon_2 + \Upsilon_4 - \frac{1}{\Gamma_{\mathrm{q}}(\beta + 1)} \right), \end{aligned} \tag{32}$$

then problem (1), (2) has at least one solution $(\mathrm{u}(\nu), \mathrm{v}(\nu))$, $\nu \in [0,1]$.

Proof. We define the number $r > 0$, which satisfies the condition

$$r \geq (\Upsilon_1 + \Upsilon_3)\|\Phi\| + (\Upsilon_2 + \Upsilon_4)\|\Psi\|, \tag{33}$$

and the closed set $\Omega_0 = \{(\mathrm{u}, \mathrm{v}) \in \mathcal{V}, \ \|(\mathrm{u}, \mathrm{v})\|_{\mathcal{V}} \leq r\}$. We shall verify the assumptions of the Krasnosel'skii fixed point theorem for the sum of two operators. We split the operator

$\mathcal{E}$ defined on Ω_0, as $\mathcal{E} = \mathcal{P} + \mathcal{Q}$, $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$, $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$, where $\mathcal{P}_i$, $\mathcal{Q}_i$, $i = 1, 2$ are defined by

$$\begin{aligned}
&\mathcal{P}_1(\mathrm{u},\mathrm{v})(\nu) = -\frac{1}{\Gamma_{\mathfrak{q}}(\alpha)} \int_0^\nu (\nu - \mathfrak{q}\tau)^{(\alpha-1)} \mathcal{F}_{\mathrm{uv}}(\tau)\, d_{\mathfrak{q}}\tau, \\
&\mathcal{Q}_1(\mathrm{u},\mathrm{v})(\nu) = \mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu) - \mathcal{P}_1(\mathrm{u},\mathrm{v})(\nu), \\
&\mathcal{P}_2(\mathrm{u},\mathrm{v})(\nu) = -\frac{1}{\Gamma_{\mathfrak{q}}(\beta)} \int_0^\nu (\nu - \mathfrak{q}\tau)^{(\beta-1)} \mathcal{G}_{\mathrm{uv}}(\tau)\, d_{\mathfrak{q}}\tau, \\
&\mathcal{Q}_2(\mathrm{u},\mathrm{v})(\nu) = \mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu) - \mathcal{P}_2(\mathrm{u},\mathrm{v})(\nu),
\end{aligned} \tag{34}$$

for all $\nu \in [0,1]$ and $(\mathrm{u},\mathrm{v}) \in \Omega_0$.

Firstly, we will prove that $\mathcal{P}(\mathrm{u}_1,\mathrm{v}_1) + \mathcal{Q}(\mathrm{u}_2,\mathrm{v}_2) \in \Omega_0$ for all $(\mathrm{u}_1,\mathrm{v}_1)$, $(\mathrm{u}_2,\mathrm{v}_2) \in \Omega_0$. For this, let $(\mathrm{u}_1,\mathrm{v}_1)$, $(\mathrm{u}_2,\mathrm{v}_2) \in \Omega_0$. Then, we find

$$\begin{aligned}
&|\mathcal{P}_1(\mathrm{u}_1,\mathrm{v}_1)(\nu) + \mathcal{Q}_1(\mathrm{u}_2,\mathrm{v}_2)(\nu)| \le \frac{1}{\Gamma_{\mathfrak{q}}(\alpha)} \int_0^\nu (\nu - \mathfrak{q}\tau)^{(\alpha-1)} |\mathcal{F}_{\mathrm{u}_1\mathrm{v}_1}(\nu)|\, d_{\mathfrak{q}}\nu \\
&\quad + \frac{\nu^{\alpha-1}}{|\Delta|} \Bigg[\frac{|\Lambda_4|}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)} \int_0^1 (1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)} |\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\, d_{\mathfrak{q}}\tau \\
&\quad + |\Lambda_4| \sum_{i=1}^{\mathfrak{a}} \frac{|a_i|}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)} \int_0^{\xi_i} (\xi_i - \mathfrak{q}\tau)^{(\alpha-\varrho_i-1)} |\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\, d_{\mathfrak{q}}\tau \\
&\quad + |\Lambda_2| \sum_{i=1}^{\mathfrak{c}} \frac{|c_i|}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)} \int_0^{\zeta_i} (\zeta_i - \mathfrak{q}\tau)^{(\alpha-\eta_i-1)} |\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\, d_{\mathfrak{q}}\tau \Bigg] \\
&\quad + \frac{\nu^{\alpha-1}}{|\Delta|} \Bigg[|\Lambda_4| \sum_{i=1}^{\mathfrak{b}} \frac{|b_i|}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)} \int_0^{\omega_i} (\omega_i - \mathfrak{q}\tau)^{(\beta-\sigma_i-1)} |\mathcal{G}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\, d_{\mathfrak{q}}\tau \\
&\quad + \frac{|\Lambda_2|}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)} \int_0^1 (1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)} |\mathcal{G}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\, d_{\mathfrak{q}}\tau \\
&\quad + |\Lambda_2| \sum_{i=1}^{\mathfrak{d}} \frac{|d_i|}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)} \int_0^{\theta_i} (\theta_i - \mathfrak{q}\tau)^{(\beta-\rho_i-1)} |\mathcal{G}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\, d_{\mathfrak{q}}\tau \Bigg] \\
&\le \|\Phi\| \Bigg\{ \frac{1}{\Gamma_{\mathfrak{q}}(\alpha)} \int_0^\nu (\nu-\mathfrak{q}\tau)^{(\alpha-1)}\, d_{\mathfrak{q}}\tau + \frac{1}{|\Delta|} \Bigg[\frac{|\Lambda_4|}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)} \int_0^1 (1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)}\, d_{\mathfrak{q}}\tau \\
&\quad + |\Lambda_4| \sum_{i=1}^{\mathfrak{a}} \frac{|a_i|}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)} \int_0^{\xi_i} (\xi_i - \mathfrak{q}\tau)^{(\alpha-\varrho_i-1)}\, d_{\mathfrak{q}}\tau \\
&\quad + |\Lambda_2| \sum_{i=1}^{\mathfrak{c}} \frac{|c_i|}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)} \int_0^{\zeta_i} (\zeta_i - \mathfrak{q}\tau)^{(\alpha-\eta_i-1)}\, d_{\mathfrak{q}}\tau \Bigg] \Bigg\} \\
&\quad + \|\Psi\| \frac{1}{|\Delta|} \Bigg[|\Lambda_4| \sum_{i=1}^{\mathfrak{b}} \frac{|b_i|}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)} \int_0^{\omega_i} (\omega_i - \mathfrak{q}\tau)^{(\beta-\sigma_i-1)}\, d_{\mathfrak{q}}\tau \\
&\quad + \frac{|\Lambda_2|}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)} \int_0^1 (1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)}\, d_{\mathfrak{q}}\tau \\
&\quad + |\Lambda_2| \sum_{i=1}^{\mathfrak{d}} \frac{|d_i|}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)} \int_0^{\theta_i} (\theta_i - \mathfrak{q}\tau)^{(\beta-\rho_i-1)}\, d_{\mathfrak{q}}\tau \Bigg] \\
&= \|\Phi\| \Bigg\{ \frac{\nu^\alpha}{\Gamma_{\mathfrak{q}}(\alpha+1)} + \frac{1}{|\Delta|} \Bigg[\frac{|\Lambda_4|}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma+1)} + |\Lambda_4| \sum_{i=1}^{\mathfrak{a}} |a_i| \frac{\xi_i^{\alpha-\varrho_i}}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i+1)} \\
&\quad + |\Lambda_2| \sum_{i=1}^{\mathfrak{c}} |c_i| \frac{\zeta_i^{\alpha-\eta_i}}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i+1)} \Bigg] \Bigg\} \\
&\quad + \|\Psi\| \frac{1}{|\Delta|} \Bigg[|\Lambda_4| \sum_{i=1}^{\mathfrak{b}} |b_i| \frac{\omega_i^{\beta-\sigma_i}}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i+1)} + |\Lambda_2| \frac{1}{\Gamma_{\mathfrak{q}}(\beta-\vartheta+1)} \\
&\quad + |\Lambda_2| \sum_{i=1}^{\mathfrak{d}} |d_i| \frac{\theta_i^{\beta-\rho_i}}{\Gamma_{\mathfrak{q}}(\beta-\rho_i+1)} \Bigg] \le \|\Phi\| \Upsilon_1 + \|\Psi\| \Upsilon_2, \ \ \forall\, \nu \in [0,1].
\end{aligned} \tag{35}$$

In a similar manner, we obtain

$$|\mathcal{P}_2(\mathrm{u}_1,\mathrm{v}_1)(\nu) + \mathcal{Q}_2(\mathrm{u}_2,\mathrm{v}_2)(\nu)| \le \|\Phi\| \Upsilon_3 + \|\Psi\| \Upsilon_4, \ \ \forall\, \nu \in [0,1]. \tag{36}$$

Therefore, we deduce

$$\begin{aligned}\|\mathcal{P}(u_1,v_1)+\mathcal{Q}(u_2,v_2)\|_{\mathcal{V}} &= \|\mathcal{P}_1(u_1,v_1)+\mathcal{Q}_1(u_2,v_2)\|+\|\mathcal{P}_2(u_1,v_1)+\mathcal{Q}_2(u_2,v_2)\| \\ &\leq (\Upsilon_1+\Upsilon_3)\|\Phi\|+(\Upsilon_2+\Upsilon_4)\|\Psi\|\leq r,\end{aligned} \tag{37}$$

that is, $\mathcal{P}(u_1,v_1)+\mathcal{Q}(u_2,v_2)\in\Omega_0$.

Subsequently, we will show that operator $\mathcal{Q}$ is a contraction mapping. Indeed, for all (u_1,v_1), $(u_2,v_2)\in\Omega_0$, by using assumption $(J2)$, we obtain

$$\begin{aligned}|\mathcal{Q}_1(u_1,v_1)(\nu)-\mathcal{Q}_1(u_2,v_2)(\nu)| &\leq C_{uv}\left(\Upsilon_1-\frac{1}{\Gamma_q(\alpha+1)}\right)+D_{uv}\Upsilon_2,\ \ \forall\nu\in[0,1],\\ |\mathcal{Q}_2(u_1,v_1)(\nu)-\mathcal{Q}_2(u_2,v_2)(\nu)| &\leq C_{uv}\Upsilon_3+D_{uv}\left(\Upsilon_4-\frac{1}{\Gamma_q(\beta+1)}\right),\ \ \forall\nu\in[0,1].\end{aligned} \tag{38}$$

Therefore, we find

$$\begin{aligned}&\|\mathcal{Q}(u_1,v_1)-\mathcal{Q}(u_2,v_2)\|_{\mathcal{V}}\\ &\leq C_{uv}\left(\Upsilon_1+\Upsilon_3-\frac{1}{\Gamma_q(\alpha+1)}\right)+D_{uv}\left(\Upsilon_2+\Upsilon_4-\frac{1}{\Gamma_q(\beta+1)}\right)\\ &=\left[\left(\mathfrak{h}_1^*+\frac{\mathfrak{h}_3^*}{\Gamma_q(\delta_1+1)}\right)\|u_1-u_2\|+\left(\mathfrak{h}_2^*+\frac{\mathfrak{h}_4^*}{\Gamma_q(\gamma_1+1)}\right)\|v_1-v_2\|\right]\\ &\quad\times\left(\Upsilon_1+\Upsilon_3-\frac{1}{\Gamma_q(\alpha+1)}\right)\\ &\quad+\left[\left(\mathfrak{k}_1^*+\frac{\mathfrak{k}_3^*}{\Gamma_q(\delta_2+1)}\right)\|u_1-u_2\|+\left(\mathfrak{k}_2^*+\frac{\mathfrak{k}_4^*}{\Gamma_q(\gamma_2+1)}\right)\|v_1-v_2\|\right]\\ &\quad\times\left(\Upsilon_2+\Upsilon_4-\frac{1}{\Gamma_q(\beta+1)}\right)\\ &=\left[\left(\mathfrak{h}_1^*+\frac{\mathfrak{h}_3^*}{\Gamma_q(\delta_1+1)}\right)\left(\Upsilon_1+\Upsilon_3-\frac{1}{\Gamma_q(\alpha+1)}\right)\right.\\ &\quad\left.+\left(\mathfrak{k}_1^*+\frac{\mathfrak{k}_3^*}{\Gamma_q(\delta_2+1)}\right)\left(\Upsilon_2+\Upsilon_4-\frac{1}{\Gamma_q(\beta+1)}\right)\right]\|u_1-u_2\|\\ &\quad+\left[\left(\mathfrak{h}_2^*+\frac{\mathfrak{h}_4^*}{\Gamma_q(\gamma_1+1)}\right)\left(\Upsilon_1+\Upsilon_3-\frac{1}{\Gamma_q(\alpha+1)}\right)\right.\\ &\quad\left.+\left(\mathfrak{k}_2^*+\frac{\mathfrak{k}_4^*}{\Gamma_q(\gamma_2+1)}\right)\left(\Upsilon_2+\Upsilon_4-\frac{1}{\Gamma_q(\beta+1)}\right)\right]\|v_1-v_2\|\\ &=\mathfrak{L}_1\|u_1-u_2\|+\mathfrak{L}_2\|v_1-v_2\|\leq\mathfrak{L}_0\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{V}}.\end{aligned} \tag{39}$$

By condition (31), we conclude that operator $\mathcal{Q}$ is a contraction.

The operators $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}$ are continuous by the continuity of functions $\mathcal{F}$ and $\mathcal{G}$. Moreover, $\mathcal{P}$ is uniformly bounded on Ω_0, because

$$\begin{aligned}\|\mathcal{P}_1(u,v)\| &\leq \frac{1}{\Gamma_q(\alpha)}\sup_{\nu\in[0,1]}\left(\int_0^\nu(\nu-q\tau)^{(\alpha-1)}|\mathcal{F}_{uv}(\tau)|\,d_q\tau\right)\\ &\leq \sup_{\nu\in[0,1]}\Phi(\nu)\,\frac{1}{\Gamma_q(\alpha)}\sup_{\nu\in[0,1]}\int_0^\nu(\nu-q\tau)^{(\alpha-1)}\,d_q\tau\\ &=\|\Phi\|\sup_{\nu\in[0,1]}\frac{\nu^\alpha}{\Gamma_q(\alpha+1)}=\frac{1}{\Gamma_q(\alpha+1)}\|\Phi\|,\ \ \forall\,(u,v)\in\Omega_0,\\ \|\mathcal{P}_2(u,v)\| &\leq \frac{1}{\Gamma_q(\beta)}\sup_{\nu\in[0,1]}\left(\int_0^\nu(\nu-q\tau)^{(\beta-1)}|\mathcal{G}_{uv}(\tau)|\,d_q\tau\right)\\ &\leq \sup_{\nu\in[0,1]}\Psi(\nu)\,\frac{1}{\Gamma_q(\beta)}\sup_{\nu\in[0,1]}\int_0^\nu(\nu-q\tau)^{(\beta-1)}\,d_q\tau\\ &=\|\Psi\|\sup_{\nu\in[0,1]}\frac{\nu^\beta}{\Gamma_q(\beta+1)}=\frac{1}{\Gamma_q(\beta+1)}\|\Psi\|,\ \ \forall\,(u,v)\in\Omega_0,\end{aligned} \tag{40}$$

and then

$$\|\mathcal{P}(\mathrm{u},\mathrm{v})\| \leq \frac{1}{\Gamma_{\mathrm{q}}(\alpha+1)}\|\Phi\| + \frac{1}{\Gamma_{\mathrm{q}}(\beta+1)}\|\Psi\|, \;\; \forall\, (\mathrm{u},\mathrm{v}) \in \Omega_0. \tag{41}$$

In the last part of the proof, we will prove that $\mathcal{P}$ is compact. Let ν_1, $\nu_2 \in [0,1]$, $\nu_1 < \nu_2$. Then for all $(\mathrm{u},\mathrm{v}) \in \Omega_0$, we obtain

$$\begin{aligned}
&|\mathcal{P}_1(\mathrm{u},\mathrm{v})(\nu_2) - \mathcal{P}_1(\mathrm{u},\mathrm{v})(\nu_1)| \\
&= \left| -\frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \int_0^{\nu_2} (\nu_2 - \mathrm{q}\tau)^{(\alpha-1)} \mathcal{F}_{\mathrm{uv}}(\tau)\, d_{\mathrm{q}}\tau + \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \int_0^{\nu_1} (\nu_1 - \mathrm{q}\tau)^{(\alpha-1)} \mathcal{F}_{\mathrm{uv}}(\tau)\, d_{\mathrm{q}}\tau \right| \\
&\leq \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \int_0^{\nu_1} \left[(\nu_2 - \mathrm{q}\tau)^{(\alpha-1)} - (\nu_1 - \mathrm{q}\tau)^{(\alpha-1)} \right] |\mathcal{F}_{\mathrm{uv}}(\tau)|\, d_{\mathrm{q}}\tau \\
&\quad + \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \int_{\nu_1}^{\nu_2} (\nu_2 - \mathrm{q}\tau)^{(\alpha-1)} |\mathcal{F}_{\mathrm{uv}}(\tau)|\, d_{\mathrm{q}}\tau \\
&\leq \|\Phi\| \left\{ \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \int_0^{\nu_1} \left[(\nu_2 - \mathrm{q}\tau)^{(\alpha-1)} - (\nu_1 - \mathrm{q}\tau)^{(\alpha-1)} \right] d_{\mathrm{q}}\tau \right. \\
&\quad \left. + \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \int_{\nu_1}^{\nu_2} (\nu_2 - \mathrm{q}\tau)^{(\alpha-1)}\, d_{\mathrm{q}}\tau \right\} \\
&= \|\Phi\| \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \left(\int_0^{\nu_2} (\nu_2 - \mathrm{q}\tau)^{(\alpha-1)}\, d_{\mathrm{q}}\tau - \int_0^{\nu_1} (\nu_1 - \mathrm{q}\tau)^{(\alpha-1)}\, d_{\mathrm{q}}\tau \right) \\
&= \|\Phi\| \frac{1}{\Gamma_{\mathrm{q}}(\alpha+1)} (\nu_2^\alpha - \nu_1^\alpha),
\end{aligned} \tag{42}$$

which tends to 0 as $\nu_2 \to \nu_1$, independently of $(\mathrm{u},\mathrm{v}) \in \Omega_0$.

In a similar manner, we find

$$|\mathcal{P}_2(\mathrm{u},\mathrm{v})(\nu_2) - \mathcal{P}_2(\mathrm{u},\mathrm{v})(\nu_1)| \leq \|\Psi\| \frac{1}{\Gamma_{\mathrm{q}}(\beta+1)} \left(\nu_2^\beta - \nu_1^\beta \right), \tag{43}$$

which tends to 0 as $\nu_2 \to \nu_1$, independently of $(\mathrm{u},\mathrm{v}) \in \Omega_0$.

Therefore, the operators $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}$ are equicontinuous. Using the Arzela–Ascoli theorem, we deduce that $\mathcal{P}$ is compact on Ω_0. Then, by the Krasnosel'skii fixed point theorem (see [33]), we conclude that problem (1), (2) has at least one solution $(\mathrm{u}(\nu),\mathrm{v}(\nu)),\ \nu \in [0,1]$. □

Theorem 3. *Suppose that* $(J1)$ *holds and the functions* $\mathcal{F},\mathcal{G} : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *are continuous and satisfy the assumptions* $(J2)$ *and* $(J3)$*. If*

$$\mathfrak{M}_0 < 1, \tag{44}$$

where $\mathfrak{M}_0 = \max\{\mathfrak{M}_1, \mathfrak{M}_2\}$ *with*

$$\begin{aligned}
\mathfrak{M}_1 &= \frac{1}{\Gamma_{\mathrm{q}}(\alpha+1)} \left(\mathfrak{h}_1^* + \frac{\mathfrak{h}_3^*}{\Gamma_{\mathrm{q}}(\delta_1+1)} \right) + \frac{1}{\Gamma_{\mathrm{q}}(\beta+1)} \left(\mathfrak{k}_1^* + \frac{\mathfrak{k}_3^*}{\Gamma_{\mathrm{q}}(\delta_2+1)} \right), \\
\mathfrak{M}_2 &= \frac{1}{\Gamma_{\mathrm{q}}(\alpha+1)} \left(\mathfrak{h}_2^* + \frac{\mathfrak{h}_4^*}{\Gamma_{\mathrm{q}}(\gamma_1+1)} \right) + \frac{1}{\Gamma_{\mathrm{q}}(\beta+1)} \left(\mathfrak{k}_2^* + \frac{\mathfrak{k}_4^*}{\Gamma_{\mathrm{q}}(\gamma_2+1)} \right),
\end{aligned} \tag{45}$$

then problem (1), (2) has at least one solution $(\mathrm{u}(\nu),\mathrm{v}(\nu)),\ \nu \in [0,1]$.

Proof. We consider again, similar to the proof of Theorem 2, the positive number $r \geq (\Upsilon_1 + \Upsilon_3)\|\Phi\| + (\Upsilon_2 + \Upsilon_4)\|\Psi\|$, and the closed set $\Omega_0 = \{(\mathrm{u},\mathrm{v}) \in \mathcal{V},\ \|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}} \leq r\}$. We also split the operator $\mathcal{E}$ defined on Ω_0 as $\mathcal{E} = \mathcal{P} + \mathcal{Q}$, $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$, $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$, where $\mathcal{P}_i$, $\mathcal{Q}_i$, $i = 1,2$ are given by (34). For $(\mathrm{u}_1,\mathrm{v}_1), (\mathrm{u}_2,\mathrm{v}_2) \in \Omega_0$, we deduce, as in the first part of the proof of Theorem 2, that $\|\mathcal{P}(\mathrm{u}_1,\mathrm{v}_1) + \mathcal{Q}(\mathrm{u}_2,\mathrm{v}_2)\|_{\mathcal{V}} \leq r$.

In what follows we will show that the operator $\mathcal{P}$ is a contraction. Indeed, for $(\mathrm{u}_1,\mathrm{v}_1)$, $(\mathrm{u}_2,\mathrm{v}_2)\in\Omega_0$, we obtain

$$\begin{aligned}
&|\mathcal{P}_1(\mathrm{u}_1,\mathrm{v}_1)(\nu)-\mathcal{P}_1(\mathrm{u}_2,\mathrm{v}_2)(\nu)|\\
&=\left|-\frac{1}{\Gamma_{\mathfrak{q}}(\alpha)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\alpha-1)}\mathcal{F}_{\mathrm{u}_1\mathrm{v}_1}(\tau)\,d_{\mathfrak{q}}\tau+\frac{1}{\Gamma_{\mathfrak{q}}(\alpha)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\alpha-1)}\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)\,d_{\mathfrak{q}}\tau\right|\\
&\le\frac{1}{\Gamma_{\mathfrak{q}}(\alpha)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\alpha-1)}|\mathcal{F}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{F}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\,d_{\mathfrak{q}}\tau\\
&\le C_{\mathrm{uv}}\frac{1}{\Gamma_{\mathfrak{q}}(\alpha)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\alpha-1)}\,d_{\mathfrak{q}}\tau=C_{\mathrm{uv}}\frac{\nu^{\alpha}}{\Gamma_{\mathfrak{q}}(\alpha+1)}\le\frac{C_{\mathrm{uv}}}{\Gamma_{\mathfrak{q}}(\alpha+1)},\ \ \forall\,\nu\in[0,1],\\
&|\mathcal{P}_2(\mathrm{u}_1,\mathrm{v}_1)(\nu)-\mathcal{P}_2(\mathrm{u}_2,\mathrm{v}_2)(\nu)|\\
&=\left|-\frac{1}{\Gamma_{\mathfrak{q}}(\beta)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\beta-1)}\mathcal{G}_{\mathrm{u}_1\mathrm{v}_1}(\tau)\,d_{\mathfrak{q}}\tau+\frac{1}{\Gamma_{\mathfrak{q}}(\beta)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\beta-1)}\mathcal{G}_{\mathrm{u}_2\mathrm{v}_2}(\tau)\,d_{\mathfrak{q}}\tau\right|\\
&\le\frac{1}{\Gamma_{\mathfrak{q}}(\beta)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\beta-1)}|\mathcal{G}_{\mathrm{u}_1\mathrm{v}_1}(\tau)-\mathcal{G}_{\mathrm{u}_2\mathrm{v}_2}(\tau)|\,d_{\mathfrak{q}}\tau\\
&\le D_{\mathrm{uv}}\frac{1}{\Gamma_{\mathfrak{q}}(\beta)}\int_0^{\nu}(\nu-\mathfrak{q}\tau)^{(\beta-1)}\,d_{\mathfrak{q}}\tau=D_{\mathrm{uv}}\frac{\nu^{\beta}}{\Gamma_{\mathfrak{q}}(\beta+1)}\le\frac{D_{\mathrm{uv}}}{\Gamma_{\mathfrak{q}}(\beta+1)},\ \ \forall\,\nu\in[0,1],
\end{aligned}\tag{46}$$

where C_{uv}, D_{uv} are given by (23).

So we conclude that

$$\begin{aligned}
&\|\mathcal{P}(\mathrm{u}_1,\mathrm{v}_1)-\mathcal{P}(\mathrm{u}_2,\mathrm{v}_2)\|_{\mathcal{V}}\\
&\le\left[\frac{1}{\Gamma_{\mathfrak{q}}(\alpha+1)}\left(\mathfrak{h}_1^*+\frac{\mathfrak{h}_3^*}{\Gamma_{\mathfrak{q}}(\delta_1+1)}\right)+\frac{1}{\Gamma_{\mathfrak{q}}(\beta+1)}\left(\mathfrak{k}_1^*+\frac{\mathfrak{k}_3^*}{\Gamma_{\mathfrak{q}}(\delta_2+1)}\right)\right]\|\mathrm{u}_1-\mathrm{u}_2\|\\
&\quad+\left[\frac{1}{\Gamma_{\mathfrak{q}}(\alpha+1)}\left(\mathfrak{h}_2^*+\frac{\mathfrak{h}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_1+1)}\right)+\frac{1}{\Gamma_{\mathfrak{q}}(\beta+1)}\left(\mathfrak{k}_2^*+\frac{\mathfrak{k}_4^*}{\Gamma_{\mathfrak{q}}(\gamma_2+1)}\right)\right]\|\mathrm{v}_1-\mathrm{v}_2\|\\
&=\mathcal{M}_1\|\mathrm{u}_1-\mathrm{u}_2\|+\mathcal{M}_2\|\mathrm{v}_1-\mathrm{v}_2\|\le\mathcal{M}_0\|(\mathrm{u}_1,\mathrm{v}_1)-(\mathrm{u}_2,\mathrm{v}_2)\|_{\mathcal{V}},
\end{aligned}\tag{47}$$

that is, by (44), the operator $\mathcal{P}$ is a contraction.

By the continuity of the functions $\mathcal{F}$ and $\mathcal{G}$, the operators $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}$ are continuous. In addition, $\mathcal{Q}$ is uniformly bounded on Ω_0, because we have

$$\begin{aligned}
|\mathcal{Q}_1(\mathrm{u},\mathrm{v})(\nu)|&\le\frac{1}{|\Delta|}\left[\frac{|\Lambda_4|}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)}\int_0^1(1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\right.\\
&\quad+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathfrak{q}\tau)^{(\alpha-\varrho_i-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\\
&\quad\left.+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathfrak{q}\tau)^{(\alpha-\eta_i-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\right]\\
&\quad+\frac{1}{|\Delta|}\left[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathfrak{q}\tau)^{(\beta-\sigma_i-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\right.\\
&\quad+\frac{|\Lambda_2|}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)}\int_0^1(1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\\
&\quad\left.+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathfrak{q}\tau)^{(\beta-\rho_i-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\right]\\
&\le\|\Phi\|\frac{1}{|\Delta|}\left[\frac{|\Lambda_4|}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma+1)}+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}|a_i|\frac{\xi_i^{\alpha-\varrho_i}}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i+1)}\right.\\
&\quad\left.+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}|c_i|\frac{\zeta_i^{\alpha-\eta_i}}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i+1)}\right]\\
&\quad+\|\Psi\|\frac{1}{|\Delta|}\left[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}|b_i|\frac{\omega_i^{\beta-\sigma_i}}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i+1)}+\frac{|\Lambda_2|}{\Gamma_{\mathfrak{q}}(\beta-\vartheta+1)}\right.\\
&\quad\left.+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}|d_i|\frac{\theta_i^{\beta-\rho_i}}{\Gamma_{\mathfrak{q}}(\beta-\rho_i+1)}\right]\\
&=\|\Phi\|\left(\Upsilon_1-\frac{1}{\Gamma_{\mathfrak{q}}(\alpha+1)}\right)+\|\Psi\|\Upsilon_2,\ \ \forall\,\nu\in[0,1],\ \ (\mathrm{u},\mathrm{v})\in\Omega_0,
\end{aligned}\tag{48}$$

and

$$
\begin{aligned}
|\mathcal{Q}_2(\mathrm{u},\mathrm{v})(\nu)| \leq & \frac{1}{|\Delta|}\Bigg[|\Lambda_1|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathfrak{q}\tau)^{(\alpha-\eta_i-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau \\
&+|\Lambda_3|\frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)}\int_0^1(1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau \\
&+|\Lambda_3|\sum_{i=1}^{\mathfrak{a}}|a_i|\frac{1}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathfrak{q}\tau)^{(\alpha-\varrho_i-1)}|\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\Bigg] \\
&+\frac{1}{|\Delta|}\Bigg[\frac{|\Lambda_1|}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)}\int_0^1(1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau \\
&+|\Lambda_1|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_{\mathfrak{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathfrak{q}\tau)^{(\beta-\rho_i-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau \\
&+|\Lambda_3|\sum_{i=1}^{\mathfrak{b}}|b_i|\frac{1}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathfrak{q}\tau)^{(\beta-\sigma_i-1)}|\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathfrak{q}}\tau\Bigg] \\
\leq & \|\Phi\|\frac{1}{|\Delta|}\Bigg[|\Lambda_1|\sum_{i=1}^{\mathfrak{c}}|c_i|\frac{\zeta_i^{\alpha-\eta_i}}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i+1)}+\frac{|\Lambda_3|}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma+1)} \\
&+|\Lambda_3|\sum_{i=1}^{\mathfrak{a}}|a_i|\frac{\xi_i^{\alpha-\varrho_i}}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i+1)}\Bigg] \\
&+\|\Psi\|\frac{1}{|\Delta|}\Bigg[\frac{|\Lambda_1|}{\Gamma_{\mathfrak{q}}(\beta-\vartheta+1)}+|\Lambda_1|\sum_{i=1}^{\mathfrak{d}}|d_i|\frac{\theta_i^{\beta-\rho_i}}{\Gamma_{\mathfrak{q}}(\beta-\rho_i+1)} \\
&+|\Lambda_3|\sum_{i=1}^{\mathfrak{b}}|b_i|\frac{\omega_i^{\beta-\sigma_i}}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i+1)}\Bigg] \\
= & \|\Phi\|\Upsilon_3+\|\Psi\|\left(\Upsilon_4-\frac{1}{\Gamma_{\mathfrak{q}}(\beta+1)}\right),\ \forall\,\nu\in[0,1],\ (\mathrm{u},\mathrm{v})\in\Omega_0.
\end{aligned} \tag{49}
$$

Therefore we deduce

$$
\begin{aligned}
\|\mathcal{Q}_1(\mathrm{u},\mathrm{v})\| &\leq \|\Phi\|\left(\Upsilon_1-\frac{1}{\Gamma_{\mathfrak{q}}(\alpha+1)}\right)+\|\Psi\|\Upsilon_2,\ \forall\,(\mathrm{u},\mathrm{v})\in\Omega_0, \\
\|\mathcal{Q}_2(\mathrm{u},\mathrm{v})\| &\leq \|\Phi\|\Upsilon_3+\|\Psi\|\left(\Upsilon_4-\frac{1}{\Gamma_{\mathfrak{q}}(\beta+1)}\right),\ \forall\,(\mathrm{u},\mathrm{v})\in\Omega_0,
\end{aligned} \tag{50}
$$

and then

$$
\|\mathcal{Q}(\mathrm{u},\mathrm{v})\|_{\mathcal{V}} \leq \|\Phi\|\left(\Upsilon_1+\Upsilon_3-\frac{1}{\Gamma_{\mathfrak{q}}(\alpha+1)}\right)+\|\Psi\|\left(\Upsilon_2+\Upsilon_4-\frac{1}{\Gamma_{\mathfrak{q}}(\beta+1)}\right),\ \forall\,(\mathrm{u},\mathrm{v})\in\Omega_0, \tag{51}
$$

that is, $\mathcal{Q}$ is uniformly bounded on Ω_0.

We finally prove that operator $\mathcal{Q}$ is compact. Let $\nu_1,\ \nu_2\in[0,1]$, $\nu_1<\nu_2$. Then for all $(\mathrm{u},\mathrm{v})\in\Omega_0$, we find

$$
\begin{aligned}
&|\mathcal{Q}_1(\mathrm{u},\mathrm{v})(\nu_2)-\mathcal{Q}_1(\mathrm{u},\mathrm{v})(\nu_1)| \\
&\leq \frac{\nu_2^{\alpha-1}-\nu_1^{\alpha-1}}{|\Delta|}\|\Phi\|\Bigg[\frac{|\Lambda_4|}{\Gamma_{\mathfrak{q}}(\alpha-\varsigma)}\int_0^1(1-\mathfrak{q}\tau)^{(\alpha-\varsigma-1)}\,d_{\mathfrak{q}}\tau \\
&\quad+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_{\mathfrak{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathfrak{q}\tau)^{(\alpha-\varrho_i-1)}\,d_{\mathfrak{q}}\tau \\
&\quad+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_{\mathfrak{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathfrak{q}\tau)^{(\alpha-\eta_i-1)}\,d_{\mathfrak{q}}\tau\Bigg] \\
&\quad+\frac{\nu_2^{\alpha-1}-\nu_1^{\alpha-1}}{|\Delta|}\|\Psi\|\Bigg[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_{\mathfrak{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathfrak{q}\tau)^{(\beta-\sigma_i-1)}\,d_{\mathfrak{q}}\tau \\
&\quad+\frac{|\Lambda_2|}{\Gamma_{\mathfrak{q}}(\beta-\vartheta)}\int_0^1(1-\mathfrak{q}\tau)^{(\beta-\vartheta-1)}\,d_{\mathfrak{q}}\tau
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_{\mathrm{q}}(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-\mathrm{q}\tau)^{(\beta-\rho_i-1)}\,d_{\mathrm{q}}\tau\Bigg]\\
&=\frac{\nu_2^{\alpha-1}-\nu_1^{\alpha-1}}{|\Delta|}\|\Phi\|\Bigg[\frac{|\Lambda_4|}{\Gamma_{\mathrm{q}}(\alpha-\varsigma+1)}+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}|a_i|\frac{\xi_i^{\alpha-\varrho_i}}{\Gamma_{\mathrm{q}}(\alpha-\varrho_i+1)}\\
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}|c_i|\frac{\zeta_i^{\alpha-\eta_i}}{\Gamma_{\mathrm{q}}(\alpha-\eta_i+1)}\Bigg]\\
&+\frac{\nu_2^{\alpha-1}-\nu_1^{\alpha-1}}{|\Delta|}\|\Psi\|\Bigg[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}|b_i|\frac{\omega_i^{\beta-\sigma_i}}{\Gamma_{\mathrm{q}}(\beta-\sigma_i+1)}+\frac{|\Lambda_2|}{\Gamma_{\mathrm{q}}(\beta-\vartheta+1)}\\
&+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}|d_i|\frac{\theta_i^{\beta-\rho_i}}{\Gamma_{\mathrm{q}}(\beta-\rho_i+1)}\Bigg]\\
&=(\nu_2^{\alpha-1}-\nu_1^{\alpha-1})\left[\|\Phi\|\left(\Upsilon_1-\frac{1}{\Gamma_{\mathrm{q}}(\alpha+1)}\right)+\|\Psi\|\Upsilon_2\right],
\end{aligned}
$$

which tends to zero as $\nu_2\to\nu_1$, independently of $(\mathrm{u},\mathrm{v})\in\Omega_0$.

In a similar manner, we obtain

$$
\begin{aligned}
&|\mathcal{Q}_2(\mathrm{u},\mathrm{v})(\nu_2)-\mathcal{Q}_2(\mathrm{u},\mathrm{v})(\nu_1)|\\
&\le(\nu_2^{\beta-1}-\nu_1^{\beta-1})\left[\|\Phi\|\Upsilon_3+\|\Psi\|\left(\Upsilon_4-\frac{1}{\Gamma_{\mathrm{q}}(\beta+1)}\right)\right],
\end{aligned}
\tag{53}
$$

which tends to zero as $\nu_2\to\nu_1$, independently of $(\mathrm{u},\mathrm{v})\in\Omega_0$.

Therefore the operators $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}$ are equicontinuous. Utilizing the Arzela–Ascoli theorem, we ascertain the compactness of $\mathcal{Q}$ on Ω_0. Consequently, employing the Krasnosel'skii fixed-point theorem, we deduce the existence of at least one solution $(\mathrm{u}(\nu),\mathrm{v}(\nu))$ $\nu\in[0,1]$ to problem (1), (2) $\square$

The forthcoming result relies on the Schaefer fixed-point theorem (refer to [34]).

Theorem 4. *Suppose that assumption $(J1)$ holds. In addition, we assume that the functions $\mathcal{F},\mathcal{G}:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ are continuous and satisfy the condition*

$(J4)$ *There exist positive constants T_1, T_2 such that*

$$
\begin{gathered}
|\mathcal{F}(\nu,\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4)|\le T_1,\ \ |\mathcal{G}(\nu,\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4)|\le T_2,\\
\forall\,\nu\in[0,1],\ \mathrm{x}_i\in\mathbb{R},\ i=1,\dots,4.
\end{gathered}
\tag{54}
$$

Then, there exists at least one solution $(\mathrm{u}(\nu),\mathrm{v}(\nu))$, $\nu\in[0,1]$ of problem (1), (2).

Proof. We prove firstly that operator $\mathcal{E}$ is completely continuous. Operator $\mathcal{E}$ is continuous. Indeed, let $(\mathrm{u}_n,\mathrm{v}_n)\in\mathcal{V}$, $n\in\mathbb{N}$, with $(\mathrm{u}_n,\mathrm{v}_n)\to(\mathrm{u},\mathrm{v})$, as $n\to\infty$ in $\mathcal{V}$. Then, for each $\nu\in[0,1]$, we deduce, as in the proof of Theorem 1, that

$$
\begin{aligned}
&|\mathcal{E}_1(\mathrm{u}_n,\mathrm{v}_n)(\nu)-\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu)|\\
&\le\frac{1}{\Gamma_{\mathrm{q}}(\alpha)}\int_0^{\nu}(\nu-\mathrm{q}\tau)^{(\alpha-1)}|\mathcal{F}_{\mathrm{u}_n\mathrm{v}_n}(\tau)-\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau\\
&\quad+\frac{1}{|\Delta|}\Bigg[\frac{|\Lambda_4|}{\Gamma_{\mathrm{q}}(\alpha-\varsigma)}\int_0^1(1-\mathrm{q}\tau)^{(\alpha-\varsigma-1)}|\mathcal{F}_{\mathrm{u}_n\mathrm{v}_n}(\tau)-\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau\\
&\quad+|\Lambda_4|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_{\mathrm{q}}(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-\mathrm{q}\tau)^{(\alpha-\varrho_i-1)}|\mathcal{F}_{\mathrm{u}_n\mathrm{v}_n}(\tau)-\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau\\
&\quad+|\Lambda_2|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_{\mathrm{q}}(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-\mathrm{q}\tau)^{(\alpha-\eta_i-1)}|\mathcal{F}_{\mathrm{u}_n\mathrm{v}_n}(\tau)-\mathcal{F}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau\Bigg]\\
&\quad+\frac{1}{|\Delta|}\Bigg[|\Lambda_4|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_{\mathrm{q}}(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-\mathrm{q}\tau)^{(\beta-\sigma_i-1)}|\mathcal{G}_{\mathrm{u}_n\mathrm{v}_n}(\tau)-\mathcal{G}_{\mathrm{uv}}(\tau)|\,d_{\mathrm{q}}\tau
\end{aligned}
\tag{55}
$$

$$+\frac{|\Lambda_2|}{\Gamma_q(\beta-\vartheta)}\int_0^1(1-q\tau)^{(\beta-\vartheta-1)}|\mathcal{G}_{u_nv_n}(\tau)-\mathcal{G}_{uv}(\tau)|\,d_q\tau$$
$$\left.+|\Lambda_2|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_q(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-q\tau)^{(\beta-\rho_i-1)}|\mathcal{G}_{u_nv_n}(\tau)-\mathcal{G}_{uv}(\tau)|\,d_q\tau\right],$$

and

$$\begin{aligned}
&|\mathcal{E}_2(u_n,v_n)(\nu)-\mathcal{E}_2(u,v)(\nu)| \\
&\le \frac{1}{|\Delta|}\Bigg[|\Lambda_1|\sum_{i=1}^{\mathfrak{c}}\frac{|c_i|}{\Gamma_q(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-q\tau)^{(\alpha-\eta_i-1)}|\mathcal{F}_{u_nv_n}(\tau)-\mathcal{F}_{uv}(\tau)|\,d_q\tau \\
&\quad+\frac{|\Lambda_3|}{\Gamma_q(\alpha-\varsigma)}\int_0^1(1-q\tau)^{(\alpha-\varsigma-1)}|\mathcal{F}_{u_nv_n}(\tau)-\mathcal{F}_{uv}(\tau)|\,d_q\tau \\
&\quad+|\Lambda_3|\sum_{i=1}^{\mathfrak{a}}\frac{|a_i|}{\Gamma_q(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-q\tau)^{(\alpha-\varrho_i-1)}|\mathcal{F}_{u_nv_n}(\tau)-\mathcal{F}_{uv}(\tau)|\,d_q\tau\Bigg] \\
&\quad+\frac{1}{\Gamma_q(\beta)}\int_0^{\nu}(\nu-q\tau)^{(\beta-1)}|\mathcal{G}_{u_nv_n}(\tau)-\mathcal{G}_{uv}(\tau)|\,d_q\tau \\
&\quad+\frac{1}{|\Delta|}\Bigg[\frac{|\Lambda_1|}{\Gamma_q(\beta-\vartheta)}\int_0^1(1-q\tau)^{(\beta-\vartheta-1)}|\mathcal{G}_{u_nv_n}(\tau)-\mathcal{G}_{uv}(\tau)|\,d_q\tau \\
&\quad+|\Lambda_1|\sum_{i=1}^{\mathfrak{d}}\frac{|d_i|}{\Gamma_q(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-q\tau)^{(\beta-\rho_i-1)}|\mathcal{G}_{u_nv_n}(\tau)-\mathcal{G}_{uv}(\tau)|\,d_q\tau \\
&\quad+|\Lambda_3|\sum_{i=1}^{\mathfrak{b}}\frac{|b_i|}{\Gamma_q(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-q\tau)^{(\beta-\sigma_i-1)}|\mathcal{G}_{u_nv_n}(\tau)-\mathcal{G}_{uv}(\tau)|\,d_q\tau\Bigg].
\end{aligned}\tag{56}$$

Because $\mathcal{F}$ and $\mathcal{G}$ are continuous, we obtain

$$\begin{aligned}
&|\mathcal{F}_{u_nv_n}(\tau)-\mathcal{F}_{uv}(\tau)|=|\mathcal{F}(\tau,u_n(\tau),v_n(\tau),I_q^{\delta_1}u_n(\tau),I_q^{\gamma_1}v_n(\tau)) \\
&\quad-\mathcal{F}(\tau,u(\tau),v(\tau),I_q^{\delta_1}u(\tau),I_q^{\gamma_1}v(\tau))|\to 0, \\
&|\mathcal{G}_{u_nv_n}(\tau)-\mathcal{G}_{uv}(\tau)|=|\mathcal{G}(\tau,u_n(\tau),v_n(\tau),I_q^{\delta_2}u_n(\tau),I_q^{\gamma_2}v_n(\tau)) \\
&\quad-\mathcal{G}(\tau,u(\tau),v(\tau),I_q^{\delta_2}u(\tau),I_q^{\gamma_2}v(\tau))|\to 0,
\end{aligned}\tag{57}$$

as $n\to\infty$, for all $\tau\in[0,1]$. Therefore, by the inequalities (55)–(57), we find

$$\|\mathcal{E}_1(u_n,v_n)-\mathcal{E}_1(u,v)\|\to 0,\ \ \|\mathcal{E}_2(u_n,v_n)-\mathcal{E}_2(u,v)\|\to 0,\ \text{ as }\ n\to\infty,\tag{58}$$

and then

$$\|\mathcal{E}(u_n,v_n)-\mathcal{E}(u,v)\|_{\mathcal{V}}\to 0,\ \text{ as }\ n\to\infty,\tag{59}$$

that is, $\mathcal{E}$ is a continuous operator.

In what follows, we prove that $\mathcal{E}$ maps bounded sets into bounded sets in $\mathcal{V}$. For $R>0$, let $\Omega_1=\{(u,v)\in\mathcal{V},\ \|(u,v)\|_{\mathcal{V}}\le R\}$. Then, by using the inequalities (54) and similar computations as those from the first part of the proof of Theorem 1, we obtain

$$|\mathcal{E}_1(u,v)(\nu)|\le T_1Y_1+T_2Y_2,\ \ |\mathcal{E}_2(u,v)(\nu)|\le T_1Y_3+T_2Y_4,\tag{60}$$

for all $\nu\in[0,1]$ and $(u,v)\in\Omega_1$. Then, we deduce

$$\|\mathcal{E}(u,v)\|_{\mathcal{V}}\le T_1(Y_1+Y_3)+T_2(Y_2+Y_4),\ \ \forall\,(u,v)\in\Omega_1,\tag{61}$$

that is, $\mathcal{E}(\Omega_1)$ is bounded.

Subsequently, we will demonstrate that $\mathcal{E}$ transforms bounded sets into equicontinuous sets. To illustrate this, consider $\nu_1,,\nu_2\in[0,1]$ with $\nu_1<\nu_2$ and $(u,v)\in\Omega_1$. Employing computations akin to those found in the proofs of Theorems 2 and 3, we arrive at the following conclusions

$$\begin{aligned}
&|\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu_2)-\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu_1)|\\
&\leq |\mathcal{P}_1(\mathrm{u},\mathrm{v})(\nu_2)-\mathcal{P}_1(\mathrm{u},\mathrm{v})(\nu_1)|+|\mathcal{Q}_1(\mathrm{u},\mathrm{v})(\nu_2)-\mathcal{Q}_1(\mathrm{u},\mathrm{v})(\nu_1)|\\
&\leq \frac{T_1}{\Gamma_{\mathrm{q}}(\alpha+1)}(\nu_2^{\alpha}-\nu_1^{\alpha})+\left[T_1\left(\Upsilon_1-\frac{1}{\Gamma_{\mathrm{q}}(\alpha+1)}\right)+T_2\Upsilon_2\right](\nu_2^{\alpha-1}-\nu_1^{\alpha-1})\to 0,\\
&|\mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu_2)-\mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu_1)|\\
&\leq |\mathcal{P}_2(\mathrm{u},\mathrm{v})(\nu_2)-\mathcal{P}_2(\mathrm{u},\mathrm{v})(\nu_1)|+|\mathcal{Q}_2(\mathrm{u},\mathrm{v})(\nu_2)-\mathcal{Q}_2(\mathrm{u},\mathrm{v})(\nu_1)|\\
&\leq \frac{T_2}{\Gamma_{\mathrm{q}}(\beta+1)}(\nu_2^{\beta}-\nu_1^{\beta})+\left[T_1\Upsilon_3+T_4\left(\Upsilon_4-\frac{1}{\Gamma_{\mathrm{q}}(\beta+1)}\right)\right](\nu_2^{\beta-1}-\nu_1^{\beta-1})\to 0,
\end{aligned} \tag{62}$$

as $\nu_2 \to \nu_1$, independently of $(\mathrm{u},\mathrm{v}) \in \Omega_2$.

Therefore, the operators $\mathcal{E}_1$ and $\mathcal{E}_2$ are equicontinuous, and so $\mathcal{E}$ is also an equicontinuous operator on Ω_2. Then, the operator $\mathcal{E}:\Omega_2 \to \mathcal{V}$ is completely continuous using the Arzela–Ascoli theorem.

In the concluding section of the proof, we establish the boundedness of the set $\mathcal{Z} = \{(\mathrm{u},\mathrm{v}) \in \mathcal{V},\ (\mathrm{u},\mathrm{v}) = \lambda\mathcal{E}(\mathrm{u},\mathrm{v}),\ 0 \leq \lambda \leq 1\}$. Take $(\mathrm{u},\mathrm{v}) \in \mathcal{V}$, implying there exists $\lambda \in [0,1]$ such that $(\mathrm{u},\mathrm{v}) = \lambda\mathcal{E}(\mathrm{u},\mathrm{v})$ or $\mathrm{u}(\nu) = \lambda\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu)$ and $\mathrm{v}(\nu) = \lambda\mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu)$ for all $\nu \in [0,1]$. Utilizing $(J4)$, we infer, similarly to the initial part of this proof, that

$$\begin{aligned}
|\mathrm{u}(\nu)| &= \lambda|\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu)| \leq |\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu)| \leq T_1\Upsilon_1+T_2\Upsilon_2,\ \ \forall\,\nu \in [0,1],\\
|\mathrm{v}(\nu)| &= \lambda|\mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu)| \leq |\mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu)| \leq T_1\Upsilon_3+T_2\Upsilon_4,\ \ \forall\,\nu \in [0,1],
\end{aligned} \tag{63}$$

and then

$$\|(\mathrm{u},\mathrm{v})|_{\mathcal{V}} = \|\mathrm{u}\|+\|\mathrm{v}\| \leq T_1(\Upsilon_1+\Upsilon_3)+T_2(\Upsilon_2+\Upsilon_4). \tag{64}$$

This final inequality indicates the boundedness of the set $\mathcal{Z}$. Consequently, employing the Schaefer fixed-point theorem, we establish the existence of at least one fixed point for the operator $\mathcal{E}$. Thus, problem (1), (2) possesses at least one solution. This concludes the proof. □

In the subsequent existence theorem, we will employ the Leray–Schauder nonlinear alternative (refer to [35]).

Theorem 5. *Assume that assumption $(J1)$ holds. In addition, we suppose that the functions $\mathcal{F},\mathcal{G}:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ are continuous and the following conditions are satisfied*

$(J5)$ *There exist the functions $\varphi_1,\ \varphi_2 \in C([0,1],\mathbb{R}_+)$ and the functions $\psi_1,\ \psi_2 \in C((\mathbb{R}_+)^4,\mathbb{R}_+)$ nondecreasing in each of variables such that*

$$\begin{aligned}
|\mathcal{F}(\nu,\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4)| &\leq \varphi_1(\nu)\psi_1(|\mathrm{x}_1|,|\mathrm{x}_2|,|\mathrm{x}_3|,|\mathrm{x}_4|),\\
|\mathcal{G}(\nu,\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4)| &\leq \varphi_2(\nu)\psi_2(|\mathrm{x}_1|,|\mathrm{x}_2|,|\mathrm{x}_3|,|\mathrm{x}_4|),
\end{aligned} \tag{65}$$

for all $\nu \in [0,1]$, $\mathrm{x}_i \in \mathbb{R},\ i=1,\dots,4$.

$(J6)$ *There exists a positive constant V such that*

$$\begin{aligned}
&\|\varphi_1\|\psi_1\left(V,V,\frac{V}{\Gamma_{\mathrm{q}}(\delta_1+1)},\frac{V}{\Gamma_{\mathrm{q}}(\gamma_1+1)}\right)(\Upsilon_1+\Upsilon_3)\\
&\quad+\|\varphi_2\|\psi_2\left(V,V,\frac{V}{\Gamma_{\mathrm{q}}(\delta_2+1)},\frac{V}{\Gamma_{\mathrm{q}}(\gamma_2+1)}\right)(\Upsilon_2+\Upsilon_4) < V.
\end{aligned} \tag{66}$$

Then, the q*-fractional boundary value problem (1), (2) has at least one solution $(\mathrm{u}(\nu),\mathrm{v}(\nu)),\ \nu \in [0,1]$.*

Proof. We define the set $\mathcal{W} = \{(\mathrm{u},\mathrm{v}) \in \mathcal{V},\ \|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}} < V\}$, where V is the constant given by (66). The operator $\mathcal{E} : \overline{\mathcal{W}} \to \mathcal{V}$ is completely continuous. We suppose that there exist $(\mathrm{u},\mathrm{v}) \in \partial\mathcal{W}$ such that $(\mathrm{u},\mathrm{v}) = \mu\mathcal{E}(\mathrm{u},\mathrm{v})$ for some $\mu \in (0,1)$. Then, we obtain

$$\begin{aligned}
&|\mathrm{u}(\nu)| = \mu|\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu)| \leq |\mathcal{E}_1(\mathrm{u},\mathrm{v})(\nu)| \\
&\leq \|\varphi_1\|\,\psi_1\left(\|\mathrm{u}\|, \|\mathrm{v}\|, \frac{\|\mathrm{u}\|}{\Gamma_{\mathrm{q}}(\delta_1+1)}, \frac{\|\mathrm{v}\|}{\Gamma_{\mathrm{q}}(\gamma_1+1)}\right)\mathrm{Y}_1 \\
&\quad +\|\varphi_2\|\,\psi_2\left(\|\mathrm{u}\|, \|\mathrm{v}\|, \frac{\|\mathrm{u}\|}{\Gamma_{\mathrm{q}}(\delta_2+1)}, \frac{\|\mathrm{v}\|}{\Gamma_{\mathrm{q}}(\gamma_2+1)}\right)\mathrm{Y}_2, \\
&|\mathrm{v}(\nu)| = \mu|\mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu)| \leq |\mathcal{E}_2(\mathrm{u},\mathrm{v})(\nu)| \\
&\leq \|\varphi_1\|\,\psi_1\left(\|\mathrm{u}\|, \|\mathrm{v}\|, \frac{\|\mathrm{u}\|}{\Gamma_{\mathrm{q}}(\delta_1+1)}, \frac{\|\mathrm{v}\|}{\Gamma_{\mathrm{q}}(\gamma_1+1)}\right)\mathrm{Y}_3 \\
&\quad +\|\varphi_2\|\,\psi_2\left(\|\mathrm{u}\|, \|\mathrm{v}\|, \frac{\|\mathrm{u}\|}{\Gamma_{\mathrm{q}}(\delta_2+1)}, \frac{\|\mathrm{v}\|}{\Gamma_{\mathrm{q}}(\gamma_2+1)}\right)\mathrm{Y}_4,
\end{aligned} \tag{67}$$

for all $\nu \in [0,1]$, and so we find

$$\begin{aligned}
&\|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}} = \|\mathrm{u}\| + \|\mathrm{v}\| \\
&\leq \|\varphi_1\|\,\psi_1\left(\|\mathrm{u}\|, \|\mathrm{v}\|, \frac{\|\mathrm{u}\|}{\Gamma_{\mathrm{q}}(\delta_1+1)}, \frac{\|\mathrm{v}\|}{\Gamma_{\mathrm{q}}(\gamma_1+1)}\right)(\mathrm{Y}_1+\mathrm{Y}_3) \\
&\quad +\|\varphi_2\|\,\psi_2\left(\|\mathrm{u}\|, \|\mathrm{v}\|, \frac{\|\mathrm{u}\|}{\Gamma_{\mathrm{q}}(\delta_2+1)}, \frac{\|\mathrm{v}\|}{\Gamma_{\mathrm{q}}(\gamma_2+1)}\right)(\mathrm{Y}_2+\mathrm{Y}_4) \\
&\leq \|\varphi_1\|\,\psi_1\left(\|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}, \|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}, \frac{\|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}}{\Gamma_{\mathrm{q}}(\delta_1+1)}, \frac{\|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}}{\Gamma_{\mathrm{q}}(\gamma_1+1)}\right)(\mathrm{Y}_1+\mathrm{Y}_3) \\
&\quad +\|\varphi_2\|\,\psi_2\left(\|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}, \|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}, \frac{\|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}}{\Gamma_{\mathrm{q}}(\delta_2+1)}, \frac{\|(\mathrm{u},\mathrm{v})\|_{\mathcal{V}}}{\Gamma_{\mathrm{q}}(\gamma_2+1)}\right)(\mathrm{Y}_2+\mathrm{Y}_4).
\end{aligned} \tag{68}$$

Therefore, we deduce

$$\begin{aligned}
V\Big/\Bigg[&\|\varphi_1\|\,\psi_1\left(V, V, \frac{V}{\Gamma_{\mathrm{q}}(\delta_1+1)}, \frac{V}{\Gamma_{\mathrm{q}}(\gamma_1+1)}\right)(\mathrm{Y}_1+\mathrm{Y}_3) \\
&+\|\varphi_2\|\,\psi_2\left(V, V, \frac{V}{\Gamma_{\mathrm{q}}(\delta_2+1)}, \frac{V}{\Gamma_{\mathrm{q}}(\gamma_2+1)}\right)(\mathrm{Y}_2+\mathrm{Y}_4)\Bigg] \leq 1,
\end{aligned} \tag{69}$$

which, by (66), is a contradiction.

We conclude that there is no $(\mathrm{u},\mathrm{v}) \in \partial\mathcal{W}$ such that $(\mathrm{u},\mathrm{v}) = \mu\mathcal{E}(\mathrm{u},\mathrm{v})$ for some $\mu \in (0,1)$. Consequently, by employing the Leray–Schauder nonlinear alternative, we infer that $\mathcal{E}$ possesses a fixed point $(\mathrm{u},\mathrm{v}) \in \overline{\mathcal{W}}$, serving as a solution to problem (1), (2). This concludes the proof. □

4. Examples

Let $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $n = 3$, $\beta = \frac{10}{3}$, $m = 4$, $\delta_1 = \frac{34}{11}$, $\gamma_1 = \frac{71}{25}$, $\delta_2 = \frac{48}{13}$, $\gamma_2 = \frac{95}{32}$, $\varsigma = \frac{6}{5}$, $\vartheta = \frac{9}{7}$, $\mathfrak{a} = \mathfrak{b} = \mathfrak{c} = \mathfrak{d} = 1$, $\varrho_1 = \frac{2}{9}$, $\sigma_1 = \frac{5}{4}$, $\eta_1 = \frac{2}{3}$, $\rho_1 = \frac{21}{10}$, $\xi_1 = \frac{1}{8}$, $\omega_1 = \frac{1}{2}$, $\zeta_1 = \frac{1}{4}$, $\theta_1 = \frac{1}{16}$, $a_1 = 3$, $b_1 = -\frac{7}{12}$, $c_1 = \frac{4}{15}$, $d_1 = -2$.

We consider the system of $\mathfrak{q}$-difference equations

$$\begin{cases} (\mathrm{D}_{1/2}^{5/2}\mathrm{u})(\nu) + \mathcal{F}\left(\nu, \mathrm{u}(\nu), \mathrm{v}(\nu), \mathrm{I}_{1/2}^{34/11}\mathrm{u}(\nu), \mathrm{I}_{1/2}^{71/25}\mathrm{v}(\nu)\right) = 0, & \nu \in (0,1), \\ (\mathrm{D}_{1/2}^{10/3}\mathrm{v})(\nu) + \mathcal{G}\left(\nu, \mathrm{u}(\nu), \mathrm{v}(\nu), \mathrm{I}_{1/2}^{48/13}\mathrm{u}(\nu), \mathrm{I}_{1/2}^{95/32}\mathrm{v}(\nu)\right) = 0, & \nu \in (0,1), \end{cases} \tag{70}$$

with the boundary conditions

$$\begin{cases} \mathrm{u}(0) = \mathrm{D}_{1/2}\mathrm{u}(0) = 0, \ \ \mathrm{D}_{1/2}^{6/5}\mathrm{u}(1) = 3\mathrm{D}_{1/2}^{2/9}\mathrm{u}\left(\frac{1}{8}\right) - \frac{7}{12}\mathrm{D}_{1/2}^{5/4}\mathrm{v}\left(\frac{1}{2}\right), \\ \mathrm{v}(0) = \mathrm{D}_{1/2}\mathrm{v}(0) = \mathrm{D}_{1/2}^{2}\mathrm{v}(0) = 0, \ \ \mathrm{D}_{1/2}^{9/7}\mathrm{v}(1) = \frac{4}{15}\mathrm{D}_{1/2}^{2/3}\mathrm{u}\left(\frac{1}{4}\right) - 2\mathrm{D}_{1/4}^{21/10}\mathrm{v}\left(\frac{1}{16}\right). \end{cases} \tag{71}$$

By using the Mathematica program, we obtain $\Lambda_1 \approx 1.05480868$, $\Lambda_2 \approx -0.48038739$, $\Lambda_3 \approx 0.10410274$, $\Lambda_4 \approx 3.76405767$, and $\Delta \approx 4.02037036 \neq 0$. Therefore, assumption $(J1)$ is satisfied.

In addition, after some computations, we find $\Upsilon_1 \approx 1.37638598$, $\Upsilon_2 \approx 0.16737542$, $\Upsilon_3 \approx 0.02789379$, and $\Upsilon_4 \approx 0.49956085$.

Example 1. *We consider the functions*

$$\begin{aligned}\mathcal{F}(\nu,\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y}) &= \frac{\sin(3\nu+2)}{\sqrt[4]{\nu^3+2}} + \frac{1}{9}e^{-(\nu-1)^2}\sqrt{\mathrm{u}^2+1} - \frac{1}{17}e^{-2\nu^3+1}\arctan \mathrm{v} \\ &\quad + \frac{\nu}{4(\nu^2+1)}\sin \mathrm{x} - \frac{3\nu}{25}\cos^2 \mathrm{y}, \\ \mathcal{G}(\nu,\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y}) &= -\nu^2+5+\frac{1}{8(\nu+1)}\sin^2 \mathrm{u} - \frac{2}{31(\nu^2+4)}\frac{|\mathrm{v}|}{1+|\mathrm{v}|} - \frac{1}{2}\mathrm{x} \\ &\quad + \frac{1}{6(\nu^3+1)}\frac{\mathrm{y}}{\mathrm{y}^2+1},\end{aligned} \tag{72}$$

for all $\nu \in [0,1]$, $\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y} \in \mathbb{R}$.

For these continuous functions, we obtain the following inequalities

$$\begin{aligned}&|\mathcal{F}(\nu,\mathrm{u}_1,\mathrm{v}_1,\mathrm{x}_1,\mathrm{y}_1) - \mathcal{F}(\nu,\mathrm{u}_2,\mathrm{v}_2,\mathrm{x}_2,\mathrm{y}_2)| \\ &\leq \mathcal{H}_1(\nu)|\mathrm{u}_1-\mathrm{u}_2| + \mathcal{H}_2(\nu)|\mathrm{v}_1-\mathrm{v}_2| + \mathcal{H}_3(\nu)|\mathrm{x}_1-\mathrm{x}_2| + \mathcal{H}_4(\nu)|\mathrm{y}_1-\mathrm{y}_2|, \\ &|\mathcal{G}(\nu,\mathrm{u}_1,\mathrm{v}_1,\mathrm{x}_1,\mathrm{y}_1) - \mathcal{G}(\nu,\mathrm{u}_2,\mathrm{v}_2,\mathrm{x}_2,\mathrm{y}_2)| \\ &\leq \mathcal{K}_1(\nu)|\mathrm{u}_1-\mathrm{u}_2| + \mathcal{K}_2(\nu)|\mathrm{v}_1-\mathrm{v}_2| + \mathcal{K}_3(\nu)|\mathrm{x}_1-\mathrm{x}_2| + \mathcal{K}_4(\nu)|\mathrm{y}_1-\mathrm{y}_2|,\end{aligned} \tag{73}$$

for all $\nu \in [0,1]$, $\mathrm{u}_i,\mathrm{v}_i,\mathrm{x}_i,\mathrm{y}_i \in \mathbb{R},\ i=1,2$, *where*

$$\begin{aligned}&\mathcal{H}_1(\nu) = \frac{1}{9}e^{-(\nu-1)^2},\ \mathcal{H}_2(\nu) = \frac{1}{17}e^{-2\nu^3+1},\ \mathcal{H}_3(\nu) = \frac{\nu}{4(\nu^2+1)},\ \mathcal{H}_4(\nu) = \frac{6\nu}{25}, \\ &\mathcal{K}_1(\nu) = \frac{1}{4(\nu+1)},\ \mathcal{K}_2(\nu) = \frac{2}{31(\nu^2+4)},\ \mathcal{K}_3(\nu) = \frac{1}{2},\ \mathcal{K}_4(\nu) = \frac{1}{6(\nu^3+1)},\end{aligned} \tag{74}$$

for all $\nu \in [0,1]$. *The functions* $\mathcal{H}_i,\ \mathcal{K}_i,\ i=1,\dots,4$ *are continuous, and we find* $\mathfrak{h}_1^* \approx 0.11111111$, $\mathfrak{h}_2^* \approx 0.15989893$, $\mathfrak{h}_3^* = 0.125$, $\mathfrak{h}_4^* = 0.24$, $\mathfrak{k}_1^* = 0.25$, $\mathfrak{k}_2^* \approx 0.01612903$, $\mathfrak{k}_3^* = 0.5$, *and* $\mathfrak{k}_4^* \approx 0.16666667$. *In addition, we obtain* $\Theta_1 \approx 0.46864611$, $\Theta_2 \approx 0.41971398$, *and so* $\Theta_0 = \Theta_1 < 1$. *Therefore, assumption* $(J2)$ *and condition (16) are satisfied. Then, by Theorem 1, we deduce that problem (70), (71) with the functions* $\mathcal{F}$ *and* $\mathcal{G}$ *given by (72) has a unique solution* $(\mathrm{u}(\nu),\mathrm{v}(\nu)),\ \nu \in [0,1]$.

Example 2. *We consider the functions*

$$\begin{aligned}\mathcal{F}(\nu,\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y}) &= \frac{\cos(\nu^3+7)}{2\nu+5} - \frac{1}{\nu^2+6}\frac{4\mathrm{u}^4+1}{\mathrm{u}^4+3} + \frac{(\nu^5+2)e^{-3\nu+1}}{6(\nu^4+27)}\sin \mathrm{v} \\ &\quad - \frac{\nu}{8(\nu^2+1)}\cos^2 \mathrm{x} + \frac{1}{\nu+4}\frac{|\mathrm{y}|}{1+3|\mathrm{y}|}, \\ \mathcal{G}(\nu,\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y}) &= -\frac{e^{-\nu+4}}{3\nu^2+1} + \frac{1}{\nu^2+7}\frac{\mathrm{u}}{\mathrm{u}^2+1} - \frac{\nu+1}{9\sqrt{\nu^4+5}}e^{-\mathrm{v}^2} \\ &\quad + \frac{2}{11}\sin^2 \mathrm{x} - \frac{\nu|\mathrm{y}|}{6(1+4|\mathrm{y}|)},\end{aligned} \tag{75}$$

for all $\nu \in [0,1]$, $\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y} \in \mathbb{R}$. *For these continuous functions, we obtain the following inequalities*

$$\begin{aligned}|\mathcal{F}(\nu,\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y})| &\leq \frac{1}{2\nu+5} + \frac{4}{\nu^2+6} + \frac{(\nu^5+2)e^{-3\nu+1}}{6(\nu^4+27)} + \frac{\nu}{8(\nu^2+1)} + \frac{1}{3(\nu+4)} =: \Phi(\nu), \\ |\mathcal{G}(\nu,\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y})| &\leq \frac{e^{-\nu+4}}{3\nu^2+1} + \frac{1}{2(\nu^2+7)} + \frac{\nu+1}{9\sqrt{\nu^4+5}} + \frac{2}{11} + \frac{\nu}{24} =: \Psi(\nu),\end{aligned} \tag{76}$$

for all $\nu \in [0,1]$, $\mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y} \in \mathbb{R}$. *In addition, we find*

$$\begin{aligned} &|\mathcal{F}(\nu, \mathrm{u}_1, \mathrm{v}_1, \mathrm{x}_1, \mathrm{y}_1) - \mathcal{F}(\nu, \mathrm{u}_2, \mathrm{v}_2, \mathrm{x}_2, \mathrm{y}_2)| \\ &\leq \mathcal{H}_1(\nu)|\mathrm{u}_1 - \mathrm{u}_2| + \mathcal{H}_2(\nu)|\mathrm{v}_1 - \mathrm{v}_2| + \mathcal{H}_3(\nu)|\mathrm{x}_1 - \mathrm{x}_2| + \mathcal{H}_4(\nu)|\mathrm{y}_1 - \mathrm{y}_2|, \\ &|\mathcal{G}(\nu, \mathrm{u}_1, \mathrm{v}_1, \mathrm{x}_1, \mathrm{y}_1) - \mathcal{G}(\nu, \mathrm{u}_2, \mathrm{v}_2, \mathrm{x}_2, \mathrm{y}_2)| \\ &\leq \mathcal{K}_1(\nu)|\mathrm{u}_1 - \mathrm{u}_2| + \mathcal{K}_2(\nu)|\mathrm{v}_1 - \mathrm{v}_2| + \mathcal{K}_3(\nu)|\mathrm{x}_1 - \mathrm{x}_2| + \mathcal{K}_4(\nu)|\mathrm{y}_1 - \mathrm{y}_2|, \end{aligned} \tag{77}$$

for all $\nu \in [0,1]$, $\mathrm{u}_i, \mathrm{v}_i, \mathrm{x}_i, \mathrm{y}_i \in \mathbb{R}$, $i = 1,2$, *where*

$$\begin{aligned} &\mathcal{H}_1(\nu) = \frac{2.9678}{\nu^2+6}, \ \mathcal{H}_2(\nu) = \frac{(\nu^5+2)e^{-3\nu+1}}{6(\nu^4+27)}, \ \mathcal{H}_3(\nu) = \frac{\nu}{4(\nu^2+1)}, \ \mathcal{H}_4(\nu) = \frac{1}{\nu+4}, \\ &\mathcal{K}_1(\nu) = \frac{1}{\nu^2+7}, \ \mathcal{K}_2(\nu) = \frac{0.8578(\nu+1)}{9\sqrt{\nu^4+5}}, \ \mathcal{K}_3(\nu) = \frac{4}{11}, \ \mathcal{K}_4(\nu) = \frac{\nu}{6}, \end{aligned} \tag{78}$$

for all $\nu \in [0,1]$. *Therefore, assumptions* $(J2)$ *and* $(J3)$ *are satisfied. In addition, we obtain* $\mathfrak{h}_1^* \approx 0.49463333$, $\mathfrak{h}_2^* \approx 0.03355903$, $\mathfrak{h}_3^* = 0.125$, $\mathfrak{h}_4^* = 0.25$, $\mathfrak{k}_1^* \approx 0.14285714$, $\mathfrak{k}_2^* \approx 0.07782119$, $\mathfrak{k}_3^* \approx 0.36363636$, $\mathfrak{k}_4^* \approx 0.16666667$, *and so* $\mathfrak{L}_1 \approx 0.56557327$, $\mathfrak{L}_2 \approx 0.25625052$, *and* $\mathfrak{L}_0 = \mathfrak{L}_1 < 1$. *Then, condition (31) is also satisfied. Therefore, by Theorem 2, we conclude that problem (70), (71) with functions (75) has at least one solution* $(\mathrm{u}(\nu), \mathrm{v}(\nu))$, $\nu \in [0,1]$.

Example 3. *We consider the functions*

$$\begin{aligned} &\mathcal{F}(\nu, \mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y}) = \frac{e^{-4\nu+3}}{\sqrt{\nu^3+1}} \cos(\mathrm{uv} + \mathrm{x}^2 - \mathrm{y}) - \frac{1}{6}\sqrt[3]{\nu^2+12}, \\ &\mathcal{G}(\nu, \mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y}) = \frac{1}{2(\nu+1)} e^{-(\mathrm{u}+\mathrm{v})^2} - \arctan\sqrt{\mathrm{x}^2+2} + \frac{\nu^2}{\nu^5+3}\,\frac{4\mathrm{y}^2+1}{\mathrm{y}^2+3} - \sin(2\nu+1), \end{aligned} \tag{79}$$

for all $\nu \in [0,1]$, $\mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y} \in \mathbb{R}$. *For these continuous functions, we obtain the following inequalities*

$$|\mathcal{F}(\nu, \mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y})| < 20.4919 = T_1, \ \ |\mathcal{G}(\nu, \mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y})| < 4.0708 = T_2. \tag{80}$$

Therefore, assumption $(J4)$ *is satisfied. Then, by Theorem 4 we infer that problem (70), (71) with functions (79) has at least one solution* $(\mathrm{u}(\nu), \mathrm{v}(\nu))$, $\nu \in [0,1]$.

Example 4. *We consider the functions*

$$\begin{aligned} &\mathcal{F}(\nu, \mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y}) = \frac{2}{\nu^3+5}\left(\frac{\mathrm{u}-3\mathrm{x}+1}{34}\sin(\mathrm{u}^2 - 8\mathrm{v} + \mathrm{x}^3) - \frac{\sqrt{\mathrm{v}^2+4\mathrm{y}^2}}{27}\arctan\frac{\mathrm{u}+\mathrm{xy}}{\mathrm{y}^2+3} + \frac{1}{9}\right), \\ &\mathcal{G}(\nu, \mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y}) = \frac{3\nu}{\nu^4+71}\left(\frac{4\mathrm{u}-5\mathrm{y}}{43}\cos(\mathrm{ux} - 3\mathrm{vy}) - \frac{\mathrm{v}+7\mathrm{x}}{11}e^{-(\mathrm{u}+3\mathrm{y})^2} - \frac{2}{15}\right), \end{aligned} \tag{81}$$

for all $\nu \in [0,1]$, $\mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y} \in \mathbb{R}$. *For these continuous functions, we obtain the inequalities*

$$\begin{aligned} &|\mathcal{F}(\nu, \mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y})| \leq \frac{2}{\nu^3+5}\left(\frac{|\mathrm{u}|}{34} + \frac{\pi|\mathrm{v}|}{54} + \frac{3|\mathrm{x}|}{34} + \frac{\pi|\mathrm{y}|}{27} + \frac{43}{306}\right), \\ &|\mathcal{G}(\nu, \mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y})| \leq \frac{3\nu}{\nu^4+71}\left(\frac{4|\mathrm{u}|}{43} + \frac{|\mathrm{v}|}{11} + \frac{7|\mathrm{x}|}{11} + \frac{5|\mathrm{y}|}{43} + \frac{2}{15}\right), \end{aligned} \tag{82}$$

for all $\nu \in [0,1]$, $\mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y} \in \mathbb{R}$. *Therefore, the continuous functions* φ_i, ψ_i, $i = 1,2$ *from (65) are given by*

$$\begin{aligned} &\varphi_1(\nu) = \frac{2}{\nu^3+5}, \ \psi_1(a,b,c,d) = \frac{a}{34} + \frac{\pi b}{54} + \frac{3c}{34} + \frac{\pi d}{27} + \frac{43}{306}, \\ &\varphi_2(\nu) = \frac{3\nu}{\nu^4+71}, \ \psi_2(a,b,c,d) = \frac{4a}{43} + \frac{b}{11} + \frac{7c}{11} + \frac{5d}{43} + \frac{2}{15}, \end{aligned} \tag{83}$$

for all $\nu \in [0,1]$ *and* $a,b,c,d \in \mathbb{R}_+$; *that is, assumption* $(J5)$ *is satisfied. In addition, we find* $\|\varphi_1\| = \frac{2}{5}$ *and* $\|\varphi_2\| = \frac{3}{72}$. *If* $V \geq 0.1$, *then assumption* $(J6)$ *is also satisfied. Therefore,* *by Theorem 5, we conclude the existence of at least one solution* $(u(\nu), v(\nu))$, $\nu \in [0,1]$ *for problem (70), (71) with functions (81).*

5. Conclusions

In this study, we explored the existence and uniqueness of solutions to a system of fractional $\mathfrak{q}$-difference equations with fractional $\mathfrak{q}$-integral terms (1), subject to the multi-point boundary conditions (2), which encompass $\mathfrak{q}$-derivatives and fractional $\mathfrak{q}$-derivatives of diverse orders. We associated an operator ($\mathcal{E}$) on the space $\mathcal{V}$ with our problem, where the solutions of (1) and (2) correspond to the fixed points of this operator. Consequently, our main results involved the utilization of various fixed-point theorems, including the Banach contraction mapping principle (employed in Theorem 1), the Krasnosel'skii fixed-point theorem for the sum of two operators (applied in Theorems 2 and 3), the Schaefer fixed-point theorem (utilized in Theorem 4), and the Leray–Schauder nonlinear alternative (employed in Theorem 5). To exemplify our findings, we concluded by presenting several illustrative examples.

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