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New Results on the Ulam–Hyers–Mittag–Leffler Stability of Caputo Fractional-Order Delay Differential Equations

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Abstract: The author considers a nonlinear Caputo fractional-order delay differential equation (CFrDDE) with multiple variable delays. First, we study the existence and uniqueness of the solutions of the CFrDDE with multiple variable delays. Second, we obtain two new results on the Ulam–Hyers–Mittag–Leffler (UHML) stability of the same equation in a closed interval using the Picard operator, Chebyshev norm, Bielecki norm and the Banach contraction principle. Finally, we present three examples to show the applications of our results. Although there is an extensive literature on the Lyapunov, Ulam and Mittag–Leffler stability of fractional differential equations (FrDEs) with and without delays, to the best of our knowledge, there are very few works on the UHML stability of FrDEs containing a delay. Thereby, considering a CFrDDE containing multiple variable delays and obtaining new results on the existence and uniqueness of the solutions and UHML stability of this kind of CFrDDE are the important aims of this work.

Keywords: delay differential equation; fractional order; Caputo fractional derivative; Ulam–Hyers–Mittag–Leffler stability

MSC: 26A33; 34K37; 45N05

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1. Introduction

The subject of the stability of various kinds of fractional differential equations (FrDEs) in the sense of Hyers–Ulam stability (HUS), Hyers–Ulam–Rassias stability (HURS), Ulam–Hyers–Mittag–Leffler stability, Lyapunov stability, etc., has gained considerable popularity and importance over the last few decades due to its extensive applications in numerous diverse and widespread fields of science and engineering (see the books by Abbas et al. [1], Balachandran [2], Benchohra et al. [3], Castro and Simões [4], Ciplea et al. [5], Diethelm [6], Hyers [7], Hyers et al. [8], Jung [9], Kilbas et al. [10], Miller and Ross [11], Podlubny [12], Saha Ray [13], Saha Ray and Sahoo [14] and Zhou [15]). In spite of the existence of numerous works dealing with the HUS, HURS, Lyapunov stability, etc., of ordinary differential equations, functional differential equations (FDEs), FrDEs, fractional differential equations, integral equations, integro-differential equations, etc., there are few works with regard to the UHML stability of these kinds of equations in the relevant literature. For example, for the HUS of FrDEs using the Laplace transform technique, see Başcı et al. [16]; for the Ulam stability of delay differential equations of fractional order using the Diaz–Margolis fixed point alternative, see Brzdek et al. [17], and the paper by Liu et al. [18], which uses the Laplace transform method; for the Ulam–Hyers–Mittag–Leffler stability of fractional-order delay difference equations using the Chebyshev norm, see Butt et al. [19], as well as the paper by Li et al. [20]; for the HU stability and HUR stability of integro-delay differential equations and integral equations using Banach's contraction theorem, see Graef et al. [21], as well as the papers by Bohner and Tunç [22], Jung [23], Tunç et al. ([24–27]), and Tunç and Tunç [28–33]; and for the HU stability of FrDEs using the Leray Schauder-type fixed-point theorem, see Khan et al. [34], as well as the paper by Liu and Li [35], which use the Laplace transform method. For some other related results, see the papers by Salim et al. [36], Shah et al. [24], Wang and Li [37], Wang et al. [38] and Wang and Zhou ([38,39]).

We should mention that the differential equation under study is stable in the sense of Ulam–Hyers stability if, for every function approximately satisfying the differential equation, there exists a solution of the equation that is close to it. This means that the Ulam–Hyers stability of a differential equation is the difference in the solutions of the inequality from those of the considered differential equation. The concept of UHML stability is suitable to describe the characteristics of fractional Ulam stability (see Wang and Li [37]).

Constituting the key reference work for this paper, in 2014, Wang and Zhang [40] investigated the existence and uniqueness of the solutions and the UHML stability of the following CFrDDE with a variable delay:

$$ {}^{C}D_t^{\alpha}x(t) = f(t, x(t), v(g(t))), t \in I \subset \mathbb{R}, \alpha \in (0,1). \tag{1}$$

In [40], the authors used Chebyshev and Bielecki norms and the Banach contraction principle to obtain the existence, uniqueness and UHML stability of solutions of the CFrDDE (1).

Motivated by the above papers, in particular by that of Wang and Zhang [40], we will consider the following CFrDDE with multiple variable delays:

$$ {}^{C}D_t^{\alpha}v(t) = \sum_{i=1}^{N} F_i(t, v(t), v(\tau_i(t))), \tag{2}$$

$$ v(t) = \phi(t), t \in [-h, 0], \tag{3}$$

where $t \in I \subset \mathbb{R}$, $I = [0, b]$, $b \in \mathbb{R}$, $b > 0$, $v(t) \in \mathbb{R}$, ${}^{C}D_t^{\alpha}v(t)$ is the Caputo fractional derivative of $v(t)$ with a lower limit zero of order α, $\alpha \in (0,1)$. We also assume that $F_i \in C(I \times \mathbb{R}^2, \mathbb{R})$, $\tau_i \in C(I, [-h, b])$, $h > 0$, $\tau_i(t) \le t$ with $0 \le \tau_i(t) \le h_i$, $h_i > 0$, $h_i \in \mathbb{R}$, $h = \max(h_i)$, $i = 1, 2, ..., N$.

It is well known that classical calculus is a special case of fractional calculus. Solving any ordinary differential equation representing a mathematical model of population dynamics or an electrical circuit gives us only one time-dependent current function; however, in the case of fractional orders, when we solve the corresponding differential equation, we obtain more than one time-dependent current function according to the value of the order. This is an important advantage in population dynamics and electronic device applications. In the same way, a fractional derivative with delay is also essential for both biologists and physicists (see, for example, the books of Balachandran [2], Diethelm [6], Kilbas et al. [10], Miller and Ross [11], Podlubny [12], Saha Ray [13], Saha Ray and Sahoo [14] and Zhou [15]).

To the best of our knowledge, there are no papers on the UHML stability of solutions of CFrDDEs with multiple variable delays in the relevant literature. Thereby, the UHML stability of the solutions of CFrDDE (2) with multiple variable delays deserves to be studied. The main aim of this paper is to extend and improve the results of Wang and Zhang [40] (Theorems 3.4, 3.5) from CFrDDE (1) with a variable delay to CFrDDE (2) with multiple variable delays and to add to the results of the papers mentioned above. In addition, we would like to provide new contributions to the theory of UHML stability. These are the essential contributions and novelty of this paper.

The remainder of this paper is structured as follows: Some basic definitions of fractional calculus theory and information are given in Section 2. The two main and new results with regard to the UHML stability of solutions to the non-linear CFrDDE (2) and three examples as numerical applications of these results are given in Section 3. Finally, at the end of the paper, a conclusion is given in Section 4.

2. Basic Information

In the present section, we provide some well-known basic definitions of fractional calculus and two lemmas which are needed in this paper.

Definition 1 (Kilbas et al. [10])**.** *A fractional integral of order γ with the lower limit of zero for a function f is defined as*

$$I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, t > 0, \ \ \gamma > 0,$$

provided the right side is point-wise defined on $\mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, where $\Gamma(.)$ is the gamma function.

Definition 2 (Kilbas et al. [10])**.** *A Riemann–Liouville derivative of order γ with the a lower limit of zero for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is described by*

$${}^{L}D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, t > 0, \ \ n-1 < \gamma < n.$$

Definition 3 (Kilbas et al. [10])**.** *The Caputo derivative of order γ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ can be written as*

$${}^{c}D_t^{\gamma} f(t) = {}^{L}D_t^{\gamma} \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), t > 0, \ \ n-1 < \gamma < n.$$

Definition 4. *Let (S, d) be a metric space. $P : S \to S$ is a Picard operator if there exists $x^* \in S$ such that*
(i) $F_P = \{x^*\}$, where $F_P = \{x \in S : P(x) = x\}$ is the fixed point set of P;
(ii) The sequence $(P^n(x_0))_{n \in N}$ converges to x^* for all $x_0 \in S$.

Lemma 1 (Rus [41])**.** *Let $(S, d, \leq)$ be an ordered metric space and $P : S \to S$ be an increasing Picard operator $\left(F_p = \{x_P^*\}\right)$. Then, for $x \in S$, $x \leq P(x)$ implies $x \geq x_P^*$.*

In this paper, we use the Henry–Gronwall inequality (see Lemma 7.1, Hyers et al. [8]), which can be used in FrDEs.

Lemma 2 (Hyers et al. [8])**.** *Let $z, \ \omega : [0, T) \to [0, \infty)$ be continuous functions, where $T \leq \infty$. If ω is nondecreasing and there are constants $\kappa \geq 0$ and $0 < \alpha < 1$ such that*

$$z(t) \leq \omega(t) + \kappa \int_0^t (t-s)^{\alpha-1} z(s) ds, t \in [0, T)$$

then

$$z(t) \leq \omega(t) + \int_0^t \left(\sum_{n=1}^{\infty} \frac{(\kappa\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \omega(s) \right) ds, t \in [0, T).$$

Remark 1. *Under the hypothesis of Lemma 2, let $\omega(t)$ be a nondecreasing function on $[0, T)$. Then, we have $z(t) \leq \omega(t) E_\alpha(\kappa\Gamma(\alpha)t^\alpha)$.*

For $F_i \in C(I \times \mathbb{R}^2, \mathbb{R})$ and $\varepsilon > 0$, we consider the initial value problem (IVP) (2) and (3) associated with the following inequality:

$$\left| {}^{C}D_t^{\alpha} \vartheta(t) - \sum_{i=1}^{N} F_i(t, \vartheta(t), \vartheta(\tau_i(t))) \right| \leq \varepsilon E_\alpha(t^\alpha), t \in I, \tag{4}$$

where E_α is defined by

$$E_\alpha(\vartheta) = \sum_{k=0}^{\infty} \frac{\vartheta^k}{\Gamma(\alpha k+1)}, \vartheta \in C, \Re(\alpha) > 0.$$

and is the so-called Mittag–Leffler function (Kilbas et al. [10]).

Definition 5. *The CFrDDE (2) is UHML stable with respect to $E_\alpha(t^\alpha)$ if there exists a positive constant M_{E_α} such that for each $\varepsilon > 0$ and for each solution $\vartheta \in C([-h,b],\mathbb{R})$ of the inequality (4) there is a solution $v \in C([-h,b],\mathbb{R})$ of the CFrDDE (2) with $|\vartheta(t) - v(t)| \leq M_{E_\alpha}\varepsilon E_\alpha(t^\alpha)$, $t \in [-h,b]$.*

Remark 2. *A function $\vartheta \in C(I,\mathbb{R})$ is a solution of the inequality (4) if and only if there exists a function $g \in C(I,\mathbb{R})$ (which depends on ϑ) such that the following conditions hold:*
(a)

$$|g(t)| \leq \varepsilon E_\alpha(t^\alpha) \text{ for all } t \in I;$$

(b)

$${}^C D_t^\alpha v(t) = \sum_{i=1}^{N} F_i(t, v(t), v(\tau_i(t))) + g(t) \text{ for all } t \in I\ .$$

Remark 3. *Let $\vartheta \in C(I,\mathbb{R})$ be a solution of the inequality (4). Then, ϑ is a solution of the below integral inequality:*

$$\begin{aligned}
&|\vartheta(t) - \vartheta(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{N} F_i(s, \vartheta(s), \vartheta(\tau_i(s))) ds| \\
&\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_\alpha(s^\alpha) ds \\
&= \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{s^{\alpha k}}{\Gamma(\alpha k+1)} ds \\
&= \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \int_0^t (t-s)^{\alpha-1} s^{\alpha k} ds \\
&= \varepsilon \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)} \\
&= \varepsilon \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\
&\leq \varepsilon E_\alpha(t^\alpha).
\end{aligned} \tag{5}$$

3. Ulam–Hyers–Mittag–Leffler Stability

We will give the first UHML stability result in Theorem 1.

Theorem 1. *Suppose that we have:*
(As1)

$$F_i \in C(I \times \mathbb{R}^2, \mathbb{R}), \tau_i \in C(I, [a-h,b]), \tau_i(t) \leq t, 0 \leq \tau_i(t) \leq h_i,$$
$$h_i > 0, h_i \in \mathbb{R}, h = \max(h_i), i = 1,2,\ldots,N.$$

(As2) There exist positive constants F_{L_i}, $i = 1, 2, \ldots, N$, such that

$$\sum_{i=1}^{N} |F_i(t, \rho_1, \rho_2(\tau_i(t))) - F_i(t, \sigma_1, \sigma_2(\tau_i(t)))|$$
$$\leq \sum_{i=1}^{N} F_{L_i}[|\rho_1 - \sigma_1| + |\rho_2(\tau_i(t)) - \sigma_2(\tau_i(t))|]$$
$$\text{for all } \; t \in I \subset \mathbb{R}, I = [0, b], b > 0, \rho_1, \rho_2, \sigma_1, \sigma_2 \in \mathbb{R}.$$

(As3)

$$\frac{2b^{\alpha}}{\Gamma(\alpha+1)} \left(\sum_{i=1}^{N} L_{F_i} \right) < 1.$$

Then,
(a) The IVPs (2) and (3) have a unique solution in $C([-h, b], \mathbb{R}) \cap C(I, \mathbb{R})$.
(b) The CFrDDE (2) is UHML stable.

Proof. The CFrDDE (2) and the initial data (3) can be converted to the following equivalent singular integral system:

$$v(t) = \begin{cases} \phi(t), \; t \in [-h, 0], \\ \phi(0) + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} \sum\limits_{i=1}^{N} F_i(s, v(s), v(\tau_i(s)))ds, \; t \in I. \end{cases} \tag{6}$$

As the first step of the proof, we will prove the existence of a solution for system (6). Hence, the singular system (6) is converted to a fixed-point problem in $E = C([-h, b], \mathbb{R})$ for an operator B_F defined by

$$B_F v(t) = \begin{cases} \phi(t), \; t \in [-h, 0], \\ \phi(0) + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} \sum\limits_{i=1}^{N} F_i(s, v(s), v(\tau_i(s)))ds, \; t \in I. \end{cases} \tag{7}$$

Here, we will show that B_F defined in (7) is a contraction mapping on $E = C([-h, b], \mathbb{R})$ with respect to the Chebyshev norm $\|.\|_{\mathbb{C}}$.

We consider $B_F : E \to E$ defined in (7). Then, for $t \in [-h, 0]$, we have

$$|B_F(v)(t) - B_F(\vartheta)(t)| = 0,$$

$$v, \; \vartheta \in C([-h, b], \mathbb{R}).$$

Next, for all $t \in I$, it follows from (As2) that

$$\begin{aligned}
&|B_F(v)(t) - B_F(\vartheta)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{N} |F_i(s, v(s), v(\tau_i(s))) - F_i(s, \vartheta(s), \vartheta(\tau_i(s)))| ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_1(s, v(s), v(\tau_1(s))) - F_1(s, \vartheta(s), \vartheta(\tau_1(s)))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_2(s, v(s), v(\tau_2(s))) - F_2(s, \vartheta(s), \vartheta(\tau_2(s)))| ds \\
&\quad + \cdots + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_N(s, v(s), v(\tau_N(s))) - F_N(s, \vartheta(s), \vartheta(\tau_N(s)))| ds \\
&\leq \frac{L_{F_1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\max_{-h \leq s \leq b} |v(s) - \vartheta(s)| + \max_{-h \leq s \leq b} |v(\tau_1(s)) - \vartheta(\tau_1(s))| \right) ds \\
&\quad + \frac{L_{F_2}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\max_{-h \leq s \leq b} |v(s) - \vartheta(s)| + \max_{-h \leq s \leq b} |v(\tau_2(s)) - \vartheta(\tau_2(s))| \right) ds \\
&\quad + \cdots + \frac{L_{F_N}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\max_{-h \leq s \leq b} |v(s) - \vartheta(s)| \right. \\
&\quad \left. + \max_{-h \leq s \leq b} |v(\tau_N(s)) - \vartheta(\tau_N(s))| \right) ds. \qquad (8)
\end{aligned}$$

Hence, using the Chebyshev $\|.\|_{\mathbb{C}}$, from (8), we get

$$\begin{aligned}
|B_F(v)(t) - B_F(\vartheta)(t)| &\leq \frac{2}{\Gamma(\alpha)} \left(\sum_{i=1}^{N} L_{F_i} \right) \|v - \vartheta\|_C \int_0^t (t-s)^{\alpha-1} ds \\
&\leq \frac{2b^{\alpha}}{\Gamma(\alpha+1)} \left(\sum_{i=1}^{N} L_{F_i} \right) \|v - \vartheta\|_C, v,\ \vartheta \in C([-h, b], \mathbb{R}).
\end{aligned}$$

Thereby, B_F is a contraction according to the Chebyshev norm $\|.\|_C$ on E. The remainder of proof can be completed via the Banach contraction principle.

We now prove our second claim of the theorem. Let $\vartheta \in C([-h, b], \mathbb{R}) \cap C(I, \mathbb{R})$ be a solution of (4). We denote by $v \in C([-h, b], \mathbb{R}) \cap C(I, \mathbb{R})$ the unique solution of the problem:

$$\begin{cases} {}^C D_t^{\alpha} v(t) = \sum_{i=1}^{N} F_i(t, v(t), v(\tau_i(t))),\ t \in I, \\ v(t) = \vartheta(t),\ t \in [-h, 0]. \end{cases}$$

Then, it follows from (As1) that

$$v(t) = \begin{cases} \vartheta(t),\ t \in [-h, 0], \\ \vartheta(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{N} F_i(s, v(s), v(\tau_i(s))) ds,\ t \in I\,. \end{cases}$$

Next, from Remark 2, we have

$$\left|\vartheta(t)-\vartheta(0)-\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}\sum_{i=1}^{N}F_i(s,\vartheta(s),\vartheta(\tau_i(s)))ds\right|\le\varepsilon E_\alpha(t^\alpha)$$

for $t\in I$. It is also obvious that $|\vartheta(t)-v(t)|=0$ for $t\in[-h,0]$.

For $t\in I$, we obtain from (As2) that

$$\begin{aligned}|\vartheta(t)-v(t)|\le&\left|\vartheta(t)-\vartheta(0)-\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}\sum_{i=1}^{N}F_i(s,\vartheta(s),\vartheta(\tau_i(s)))ds\right|\\&+\left|\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}\sum_{i=1}^{N}[F_i(s,\vartheta(s),\vartheta(\tau_i(s)))-F_i(s,v(s),v(\tau_i(s)))]ds\right|\\\le&\varepsilon E_\alpha(t^\alpha)+\frac{L_{F_1}}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}(|\vartheta(s)-v(s)|+|\vartheta(\tau_1(s))-v(\tau_1(s))|)ds\\&+\frac{L_{F_2}}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}(|\vartheta(s)-v(s)|+|\vartheta(\tau_2(s))-v(\tau_2(s))|)ds\\&+\cdots+\frac{L_{F_N}}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}(|\vartheta(s)-v(s)|+|\vartheta(\tau_N(s))-v(\tau_N(s))|)ds\\=&\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}\left(\sum_{i=1}^{N}L_{F_i}(|\vartheta(s)-v(s)|+|\vartheta(\tau_i(s))-v(\tau_i(s))|)\right)ds.\end{aligned}\tag{9}$$

For $\omega\in C([-h,b],\mathbb{R}^+)$, we consider the operator

$$T:C([-h,b],\mathbb{R}^+)\to C([-h,b],\mathbb{R}^+)$$

defined by

$$T(\omega)(t)=\begin{cases}0,\ t\in[-h,0],\\\varepsilon E_\alpha(t^\alpha)+\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}\sum\limits_{i=1}^{N}L_{F_i}(\omega(s)+\omega(\tau(s_i)))ds,\ t\in I\,.\end{cases}$$

Next, we will verify that T is a Picard operator. For all $t\in I$, from (As2), it follows that

$$\begin{aligned}|T(\omega)(t)-T(\ell)(t)|\le&\frac{L_{F_1}}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}(|\omega(s)-\ell(s)|+|\omega(\tau_1(s))-\ell(\tau_1(s))|)ds\\&+\frac{L_{F_2}}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}(|\omega(s)-\ell(s)|+|\omega(\tau_2(s))-\ell(\tau_2(s))|)ds\\&+\cdots+\frac{L_{F_N}}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}(|\omega(s)-\ell(s)|+|\omega(\tau_N(s))-\ell(\tau_N(s))|)ds\\\le&\frac{2b^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{N}L_{F_i}\right)\|\omega-\ell\|_{\mathbb{C}}\end{aligned}$$

for all $\omega,\ \ell \in C([a-h,b],\mathbb{R}^+)$. Thereby, we obtain

$$\|T(\omega)-T(\ell)\|_{\mathbb{C}} \le \frac{2b^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{N} L_{F_i}\right)\|\omega-\ell\|_{\mathbb{C}},$$

for all $\omega,\ \ell \in C([a-h,b],\mathbb{R}^+)$. Hence, we conclude that T is a contraction via the Chebyshev norm $\|.\|_{\mathbb{C}}$ on $C([a-h,b],\mathbb{R}^+)$ according to (As3).

We will now apply the Banach contraction principle to the operator T such that T is a Picard operator and $\mathrm{F}_T = \{\omega^*\}$.

Hence, for $t \in I$, we have

$$\omega^*(t) = \varepsilon E_{\alpha}(t^{\alpha}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sum_{i=1}^{N}\left(L_{F_i}\right)(\omega^*(s)+\omega^*(\tau_i(s)))ds.$$

We will now show that the solution ω^* is increasing. For $0 \le t_1 < t_2 \le b$, let

$$m = \min_{s\in I}\sum_{i=1}^{N}(\omega^*(s)+\omega^*(\tau_i(s))) \in \mathbb{R}^+.$$

Then, we have

$$\begin{aligned}
\omega^*(t_2)-\omega^*(t_1) =& \varepsilon[E_{\alpha}(t_2^{\alpha})-E_{\alpha}(t_1^{\alpha})] \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\sum_{i=1}^{N} L_{F_i}(\omega^*(s)+\omega^*(\tau_i(s)))ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\sum_{i=1}^{N} L_{F_i}(\omega^*(s)+\omega^*(\tau_i(s)))ds
\end{aligned}$$

$$\begin{aligned}
\ge& \varepsilon[E_{\alpha}(t_2^{\alpha})-E_{\alpha}(t_1^{\alpha})] + \frac{m}{\Gamma(\alpha)}\left(\sum_{i=1}^{N} L_{F_i}\right)\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds \\
&+ \frac{m}{\Gamma(\alpha)}\left(\sum_{i=1}^{N} L_{F_i}\right)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
=& \varepsilon[E_{\alpha}(t_2^{\alpha})-E_{\alpha}(t_1^{\alpha})] + \frac{m}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{N} L_{F_i}\right)(t_2^{\alpha}-t_1^{\alpha}) > 0.
\end{aligned}$$

Thereby, ω^* is increasing. Hence, we have $\omega^*(\tau_i(t)) \le \omega^*(t)$ due to $\tau_i(t) \le t$, $i = 1,2,\ldots,N$, and

$$\omega^*(t) \le \varepsilon E_{\alpha}(t^{\alpha}) + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sum_{i=1}^{N}\left(L_{F_i}\right)\omega^*(s)ds.$$

Using Lemma 2 and Remark 1, we get

$$\omega^*(t) \le \mathrm{M}_{E_{\alpha}}\varepsilon E_{\alpha}(t^{\alpha}), t \in [-h,b],$$

where $M_{E_{\alpha}} = E_{\alpha}\left(2b^{\alpha}\sum_{i=1}^{N}\left(L_{F_i}\right)\right)$.

In particular, if $\omega = |\vartheta - v|$, then from (9), we have $\omega \le T\omega$. Next, applying Lemma 2, we get $\omega \le \omega^*$, where T is a Picard operator and it is also an increasing operator. As a result, we conclude that

$$|\vartheta(t) - v(t)| \le \mathrm{M}_{E_\alpha} \varepsilon E_\alpha(t^\alpha), t \in [-h, b].$$

Thus, the CFrDDE (2) is UHML stable. □

We present the first example to show the application of Theorem 1 in a particular case of the CFrDDE (2).

Example 1. *Consider the following CFrDDE containing a constant delay:*

$$\begin{cases} {}^C D_t^{\frac{1}{2}} v(t) = \frac{1}{40} v(t) + + \frac{1}{20} \sin(v(t)) + \frac{1}{40} \frac{\sin(v(t-1))}{1+t^4}, \ t \in [0,1], \\ v(0) = 0, \ t \in [-1, 0] \end{cases} \tag{10}$$

and the inequality

$$\left| {}^C D_t^{\frac{1}{2}} \vartheta(t) - F_1(t, \vartheta(t), \vartheta(t-1)) \right| \le \varepsilon E_{\frac{1}{2}}(t^{\frac{1}{2}}).$$

We note that CFrDDE (10) is in the form of CFrDDE (2) with the data as follows:

$$\alpha = \frac{1}{2}, [0, b] = [0, 1], [-h, 0] = [-1, 0],$$

$$0 < \tau_1(t) = 1 = h_1, h = h_1 = 1 > 0, h_1 \in \mathbb{R},$$

$$F_1(t, \vartheta, \vartheta(\tau_1(t))) = \frac{1}{40} v + \frac{1}{20} \sin(v) + \frac{1}{40} \frac{\sin(v(t-1))}{1+t^4}.$$

We will now show that the conditions (As1),(As3) and (As3) of Theorem 1 hold. It is clear that $F_1 \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$. Thus, (As1) holds.

We now let $L_{F_1} = \frac{3}{40}$, $\alpha = 2^{-1}$, $I = [0,1] = [0,b]$ and calculate

$$\begin{aligned} &|F_1(t, \rho_1, \rho_2(\tau_1(t))) - F_i(t, \sigma_1, \sigma_2(\tau_1(t)))| \\ &\le \frac{1}{40} |\rho_1 - \sigma_1| + \frac{1}{20} |\sin(\rho_1) - \sin(\sigma_1)| \\ &\quad + \frac{1}{40(1+t^4)} |\sin(\rho_2(t-1)) - \sin(\sigma_2(t-1))| \\ &\le \frac{1}{40} |\rho_1 - \sigma_1| + \frac{1}{20} |\rho_1 - \sigma_1| + \frac{1}{40} |\sin(\rho_2(t-1)) - \sin(\sigma_2(t-1))| \\ &\le \frac{3}{40} |\rho_1 - \sigma_1| + \frac{1}{40} |\rho_2(t-1) - \sigma_2(t-1)| \\ &\le \frac{3}{40} [|\rho_1 - \sigma_1| + |\rho_2(t-1) - \sigma_2(t-1)|] \end{aligned}$$

and

$$\frac{2b^\alpha L_{F_1}}{\Gamma(\alpha+1)} = \frac{6}{40\Gamma(\frac{3}{2})} = \frac{3}{10\sqrt{\pi}} = \frac{3}{17,772453} < 1.$$

Thereby, conditions (As2) and (As3) of Theorem 1 hold. Then, the IVP (10) has a unique solution and the CFrDDE in (10) is UHML stable with

$$|\vartheta(t) - v(t)| \le M_{E_{2^{-1}}} \varepsilon E_{2^{-1}}(t^{2^{-1}}), t \in [-1, 1],$$

where $M_{E_{2^{-1}}} = E_\alpha(2b^\alpha L_{F_1}) = E_{2^{-1}}(\frac{3}{20})$. So, the application of Theorem 1 is provided by Example 1.

We present the second example to verify the numerical application of Theorem 1 in a particular case of the CFrDDE (2).

Example 2. *Consider the following CFrDDE containing a variable delay:*

$$\begin{cases} {}^{C}D_t^{\frac{1}{2}} v(t) = \frac{1}{200} v^2(t) + \frac{1}{500+t^2} \sin(v(t)) + \frac{1}{100} \sin(v(t/2)), \ t \in [0,1], \\ v(0) = 0, \ t \in [-1,0] \end{cases} \tag{11}$$

and the inequality

$$\left| {}^{C}D_t^{\frac{1}{2}} \vartheta(t) - F_1(t, \vartheta(t), \vartheta(t/2)) \right| \le \varepsilon E_{\frac{1}{2}}(t^{\frac{1}{2}}).$$

We note that CFrDDE (11) is in the form of CFrDDE (2) with the data as follows:

$$\alpha = \frac{1}{2}, [0,b] = [0,1], [-h,0] = [-1,0],$$

$$\tau_1(t) = \frac{t}{2}, 0 \le \tau_1(t) = \frac{t}{2} \le t \le 1, \forall t \in [0,1],$$

$$h_1 = h = 1, h_1 \in \mathbb{R},$$

$$F_1(t, \vartheta, \vartheta(\tau_1(t))) = \frac{1}{200} v^2 + \frac{1}{500+t^2} \sin(v) + \frac{1}{100} \sin(v(t/2)).$$

We will now verify that the conditions (As1), (As3) and (As3) of Theorem 1 hold. It is clear that $F_1 \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$.

Thus, (As1) holds. We now choose $L_{F_1} = \frac{3}{250}$, $\alpha = 2^{-1}$, $I = [0,1] = [0,b]$ and calculate

$$\begin{aligned} &|F_1(t, \rho_1, \rho_2(\tau_1(t))) - F_1(t, \sigma_1, \sigma_2(\tau_1(t)))| \\ &\quad \le \frac{1}{200} |\rho_1 + \sigma_1| |\rho_1 - \sigma_1| + \frac{1}{500} |\sin(\rho_1) - \sin(\sigma_1)| \\ &\quad\quad + \frac{1}{100} |\sin(\rho_2(t/2)) - \sin(\sigma_2(t/2))| \\ &\quad \le \frac{1}{100} |\rho_1 - \sigma_1| + \frac{1}{500} |\rho_1 - \sigma_1| + \frac{1}{100} |\rho_2(t/2) - \sigma_2(t/2)| \\ &\quad = \frac{3}{250} |\rho_1 - \sigma_1| + \frac{1}{100} |\rho_2(t/2) - \sigma_2(t/2)| \\ &\quad \le \frac{3}{250} [|\rho_1 - \sigma_1| + |\rho_2(t/2) - \sigma_2(t/2)|] \end{aligned}$$

$$\frac{2b^{\alpha} L_{F_1}}{\Gamma(\alpha+1)} = \frac{3}{125\Gamma\left(\frac{3}{2}\right)} = \frac{6}{125\sqrt{\pi}} = \frac{6}{221,601315} < 1.$$

Hence, conditions (As2) and (As3) of Theorem 1 hold. Then, the IVP (11) has a unique solution and the CFrDDE in (11) admits UHML stability.

Next, we use Bielecki's norm $\|.\|_B$, i.e., $\|x\|_B = \max_{t \in J} |x(t)| \exp(-\theta t), \theta > 0, J \subset \mathbb{R}^+$, to derive similar results to the above for CFrDDE (2).

We will give the second UHML stability result in Theorem 2.

Theorem 2. *Suppose that we have (As1), (As2) of Theorem 1 and*
(As4)

$$\frac{2 \sum\limits_{i=1}^{N} (L_{F_i})}{\Gamma(\alpha)} \times \frac{b^{\alpha} \exp(\theta b)}{\sqrt{2(2\alpha - 1)\theta}} < 1 \text{ for some } \alpha \in (2^{-1}, 1) \text{ and } \theta > 0.$$

(a) The IVPs (2) and (3) have a unique solution in $C([-h,b], \mathbb{R}) \cap C(I, \mathbb{R})$.
(b) The CFrDDE (2) is UHML stable.

Proof. Just like the discussion in Theorem 1, we will only prove that B_F, defined as before in Theorem 1, is a contraction on E via Bielecki's norm $\|.\|_B$. Since the process is standard, we only give the main difference in the proof as follows:

For all $t \in I$, it follows from (As2) that

$$\begin{aligned}
&|B_F(v)(t) - B_F(\vartheta)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{N} |F_i(s, v(s), v(\tau_i(s))) - F_i(s, \vartheta(s), \vartheta(\tau_i(s)))| ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_1(s, v(s), v(\tau_1(s))) - F_1(s, \vartheta(s), \vartheta(\tau_1(s)))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_2(s, v(s), v(\tau_2(s))) - F_2(s, \vartheta(s), \vartheta(\tau_2(s)))| ds \\
&\quad + \cdots + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_N(s, v(s), v(\tau_N(s))) - F_N(s, \vartheta(s), \vartheta(\tau_N(s)))| ds \\
&\leq \frac{L_{F_1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\theta s} \left(\max_{-h \leq s \leq b} |v(s) - \vartheta(s)| e^{-\theta s} + \max_{-h \leq s \leq b} |v(\tau_1(s)) - \vartheta(\tau_1(s))| e^{-\theta s} \right) ds \\
&+ \frac{L_{F_2}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\theta s} \left(\max_{-h \leq s \leq b} |v(s) - \vartheta(s)| e^{-\theta s} + \max_{-h \leq s \leq b} |v(\tau_2(s)) - \vartheta(\tau_2(s))| e^{-\theta s} \right) ds \\
&+ \cdots + \frac{L_{F_N}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\theta s} \left(\max_{-h \leq s \leq b} |v(s) - \vartheta(s)| e^{-\theta s} \right. \\
&+ \left. \max_{-h \leq s \leq b} |v(\tau_N(s)) - \vartheta(\tau_N(s))| e^{-\theta s} \right) ds.
\end{aligned} \tag{12}$$

Hence, using the Chebyshev $\|.\|_B$ norm, from (12), we get

$$|B_F(v)(t) - B_F(\vartheta)(t)| \leq \frac{2}{\Gamma(\alpha)} \left(\sum_{i=1}^{N} L_{F_i} \right) \|v - \vartheta\|_B \int_0^t (t-s)^{\alpha-1} e^{\theta s} ds.$$

For the next step, we note that for $\alpha \in (2^{-1}, 1)$ and the Hölder's inequality:

$$\begin{aligned}
\int_0^t (t-s)^{\alpha-1} e^{\theta s} ds &\leq \left(\sqrt{\int_0^t (t-s)^{2(\alpha-1)} e^{\theta s} ds} \right) \times \left(\sqrt{\int_0^t e^{2\theta s} ds} \right) \\
&\leq \frac{1}{\sqrt{2\theta}} \times \frac{b^\alpha}{\sqrt{2\alpha - 1}} \exp(\theta b).
\end{aligned}$$

Then, we obtain

$$\|B_F(v) - B_F(\vartheta)\|_B \leq \frac{2 \sum_{i=1}^{N} (L_{F_i})}{\Gamma(\alpha)} \times \frac{b^\alpha \exp(\theta b)}{\sqrt{2(2\alpha - 1)\theta}} \|v - \vartheta\|_B$$

for all $v,\ \vartheta \in C([-h,b],\mathbb{R}^+)$. Hence, B_F is a contraction due to (As4) via Bielecki's norm $\|.\|_B$ on $C([-h,b],\mathbb{R}^+)$. The proof of UHML stability is the same as in Theorem 1. Thereby, we omit the rest of the proof. □

We present a third example to show the application of Theorem 2 in the particular case of CFrDDE (2).

Example 3. *Consider the following CFrDDE containing a constant delay:*

$$\begin{cases} {}^{C}D_t^{\frac{2}{3}}v(t) = \frac{1}{50}v(t) + \frac{1}{100}\exp(-t^2)cos(v(t)) + \frac{1}{50}\sin\big(v(t-4^{-1})\big),\ t \in \Big[0,4^{-1}\Big], \\ v(0) = 0,\ t \in [-4^{-1},0] \end{cases} \tag{13}$$

and the inequality

$$\left| {}^{C}D_t^{\frac{2}{3}}\vartheta(t) - F_1(t,\vartheta(t),\vartheta(t-4^{-1}))\right| \le \varepsilon E_{\frac{2}{3}}(t^{\frac{2}{3}}).$$

We note that CFrDDE (13) is in the form of CFrDDE (2) with the data as follows:

$$\alpha = \frac{2}{3}, [0,b] = [0,4^{-1}], [-h,0] = [-4^{-1},0],$$

$$F_1(t,\vartheta,\vartheta(\tau_1(t))) = \frac{1}{50}v + \frac{1}{100}\exp(-t^2)cos(v) + \frac{1}{50}\sin\Big(v(t-4^{-1})\Big).$$

We will now show that the conditions (As1), (As2) and (As4) of Theorem 2 hold. For this aim, we let $L_{F_1} = \frac{3}{100}$, $\alpha = \frac{2}{3}$, $\theta = \frac{1}{2}$, $I = [0,4^{-1}] = [0,b]$ and calculate

$$\begin{aligned} &|F_1(t,\rho_1,\rho_2(\tau_1(t))) - F_i(t,\sigma_1,\sigma_2(\tau_1(t)))| \\ &\quad \le \frac{1}{50}|\rho_1-\sigma_1| + \frac{1}{100}\exp(-t^2)|\cos(\rho_1)-\cos(\sigma_1)| \\ &\qquad + \frac{1}{50}\Big|\sin\Big(\rho_2(t-4^{-1})\Big) - \sin\Big(\sigma_2(t-4^{-1})\Big)\Big| \\ &\quad \le \frac{1}{50}|\rho_1-\sigma_1| + \frac{1}{100}|\rho_1-\sigma_1| + \frac{1}{50}\Big|\rho_2(t-4^{-1}) - \sigma_2(t-4^{-1})\Big| \\ &\quad = \frac{3}{100}|\rho_1-\sigma_1| + \frac{1}{50}\Big|\rho_2(t-4^{-1}) - \sigma_2(t-4^{-1})\Big| \\ &\quad \le \frac{3}{100}\Big[|\rho_1-\sigma_1| + \Big|\rho_2(t-4^{-1}) - \sigma_2(t-4^{-1})\Big|\Big] \end{aligned}$$

and

$$\frac{2L_{F_1}}{\Gamma(\alpha)} \times \frac{b^{\alpha}\exp(\theta b)}{\sqrt{2(2\alpha-1)\theta}} = \frac{3}{50\Gamma\left(\frac{2}{3}\right)} \times \frac{\left(\frac{1}{4}\right)^{\frac{2}{3}}\exp(8^{-1})}{\frac{1}{\sqrt{3}}} = 0.0348 < 1.$$

Thereby, the conditions (As1), (As2) and (As4) of Theorem 2 hold. Then, IVP (13) has a unique solution and the CFrDDE in (13) is UHML stable with

$$|\vartheta(t) - v(t)| \le M_{E_{\frac{2}{3}}}\varepsilon E_{\frac{2}{3}}(t^{\frac{2}{3}}), t \in [-4^{-1},4^{-1}],$$

where $M_{E_\alpha} = E_\alpha\big(2b^{\alpha}L_{F_1}\big) = E_{\frac{2}{3}}\left[\frac{3}{50}\left(\frac{1}{4}\right)^{\frac{2}{3}}\right]$. So, the application of Theorem 2 is provided by Example 3.

Remark 4. *As mentioned above, to prove the main results of this paper, we benefit from the Banach contraction principle via the Chebyshev norm, the Bielecki norm, the Picard operator and Mittag–Leffler functions. Indeed, the Banach contraction principle is well known and in the past and up to today it has been used very effectively in various research fields in the relevant literature. To the best* *of our knowledge, in this paper, the innovativeness with regard to this fixed-point method is that, for the first time, we have applied this method to new mathematical models as a CFrDDE containing multiple variable delays via Mittag–Leffler functions and different norms. Mittag–Leffler functions are very suitable for investigating an Ulam-type stability called UHML stability. Hence, this paper has novelty and presents new results.*

Remark 5. *The differences between Theorem 1 and Theorem 2 are outlined as follows: Theorem 1 has been proven via the Banach contraction principle, the Chebyshev norm $\|.\|_{\mathbb{C}}$, which is defined as*

$$\|\vartheta\|_{\mathbb{C}} := \max_{t\in J} |\vartheta(t)|, \ J \subset \mathbb{R}^+, \ \mathbb{R}^+ = [0,\infty),$$

and the Picard operator. However, Theorem 2 has been proven via the Banach contraction principle, the Bielecki norm $\|.\|_B$, which is defined by

$$\|x\|_B = \max_{t\in J} |x(t)| \exp(-\theta t), \theta > 0, \theta \in \mathbb{R},$$

and the Picard operator. As it is seen, the definitions of the Chebyshev norm $\|.\|_{\mathbb{C}}$ and the Bielecki norm $\|.\|_B$ are different. Hence, conditions (As3) of Theorem 1 and (As4) of Theorem 2 are different. Furthermore, in the relevant literature, using the Chebyshev norm and Bielecki norm separately to prove qualitative results of the same mathematical model is an interesting and usual method (see, for example, Kh.Niazi et al. [42] and Wang and Zhang [40]).

Remark 6. *The main results of this paper (Theorem 1 and Theorem 2) have sufficient conditions for the IVPs* (2)*, and* (3) *to have a unique solution and for the UHML stability of CFrDDE* (2)*. To the best of our knowledge, the sufficient conditions for UHML stability to work well depend on the mathematical model(s) under study and the technique(s) used in the proof(s). We believe that these have been presented in this article.*

4. Conclusions

This work dealt with qualitative behaviors of a nonlinear CFrDDE involving multiple variable delays. In the present work, we constructed new sufficient conditions with regard to the qualitative behaviors, namely the existence of solutions, the uniqueness of solutions and the UHML stability of the considered nonlinear CFrDDE involving multiple variable delays. We proved two new results with regard to these qualitative concepts depending upon sufficient conditions. The approach used in the proofs is based on the fixed-point method using the Picard operator and the Bielecki norm. In two particular cases, applications of the new results were provided via three numerical examples. To the best of our knowledge, there are no results in the relevant literature on the UHML stability of nonlinear CFrDDEs involving multiple variable delays. The results of this work have scientific novelty and complement the results found in the relevant literature. Based on the present work, there are several potential directions for further research related to the UHML stability of Caputo fractional delay integro-differential equations and Riemann–Liouville fractional delay integro-differential equations as open problems.

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