Supplementary document file

Reversible transformation of a vesicular aggregate in response to pH oscillation

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Analytical solution of Eq. 1

The solutions to Eq. 1 for conditions (i), (ii), (i'), and (ii') are as follows.

$$\frac{dx}{dt} = a\{f(t) - x\} \quad (1)$$

For (i) and (ii), Laplace transform may be applied.

$$X(s) = \frac{x(0)}{s+a} + \frac{a}{s+a}F(s),$$

where $X(s)$ and $F(s)$ are the Laplace transforms of $x(t)$ and $f(t)$, respectively.

(i) $f(t) = 0,\ t < 0$ and $t > \tau;\ \ = K,\ 0 < t < \tau$

$$F(s) = \int_0^\tau Ke^{-st}\,dt = \frac{K(1 - e^{-s\tau})}{s}$$

As $x(0) = 0$, the inverse Laplace transform provides

$$x(t) = Ka\left[(1 - e^{-at}) - H(t - \tau)\left\{1 - e^{-a(t-\tau)}\right\}\right].$$

Here, $H(t)$ denotes the Heaviside function.

(ii) $f(t) = 0,\ t < 0$ and $t > \tau;\ \ = -k(t - \tau),\ 0 < t < \tau$

$$F(s) = -k\int_0^\tau (t - \tau)e^{-st}\,dt = -\frac{k(1 - e^{-s\tau})}{s^2} + \frac{k\tau}{s}$$

As $x(0) = 0$, the inverse Laplace transform provides

$$x(t) = \frac{k(1+a\tau)}{a}(1 - e^{-at}) - \frac{k}{a}H(t - \tau)\left(1 - e^{-a(t-\tau)}\right) - k\{t - R(t - \tau)\}.$$

Here, $R(t)$ denotes the Ramp function

$$R(t) = t \text{ at } t > 0;\ = 0 \text{ at } t < 0.$$

For (i') and (ii'), Fourier series may be applied.

(i') $f(t) = 0$ at $-\tau < t < -\frac{\tau}{2}$ and $\frac{\tau}{2} < t < \tau,;\ \ = K$ at $-\frac{\tau}{2} < t < \frac{\tau}{2}$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi t}{\tau}},\ \ f(t) = \sum_{n=-\infty}^{\infty} f_n e^{i\frac{n\pi t}{\tau}},$$

where c_n and f_n are complex Fourier coefficients of $x(t)$ and $f(t)$, respectively.

Applying the Fourier series to Eq. 1 yields

$$c_n = \frac{af_n}{a+i\frac{n\pi}{\tau}}.$$

Here,

$$c_n = \frac{1}{2\tau}\int_{-\tau}^{\tau} x(t)e^{-i\frac{n\pi t}{\tau}}dt\,, \qquad f_n = \frac{1}{2\tau}\int_{-\tau}^{\tau} f(t)e^{-i\frac{n\pi t}{\tau}}dt$$

These yield

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{a}{a+i\frac{n\pi}{\tau}} \frac{K}{n\pi} \sin\frac{n\pi}{2} e^{i\frac{n\pi t}{\tau}}.$$

Using trigonometric function, $x(t)$ becomes

$$x(t) = \frac{K}{2} + \sum_{n=1}^{\infty} \frac{K}{n\pi} \sin\frac{n\pi}{2} \frac{2a}{a^2+\left(\frac{n\pi}{\tau}\right)^2}\left(a\cos\frac{n\pi t}{\tau} + \frac{n\pi}{\tau}\sin\frac{n\pi t}{\tau}\right).$$

(ii') $f(t) = 0$ at $-\tau < t < 0$; $= -k(t-\tau)\ 0 < t < \tau$

The same procedure for the given $f(t)$ yields

$$x(t) = \frac{k\tau}{4} + \frac{1}{2\tau}\sum_{n=1}^{\infty} \frac{a}{a^2+\left(\frac{n\pi}{\tau}\right)^2}\left[\frac{2k\tau^2}{n\pi}\left\{a\sin\frac{n\pi t}{\tau} - \frac{n\pi}{\tau}\cos\frac{n\pi t}{\tau}\right\} - k\left(\frac{\tau}{n\pi}\right)^2(\cos n\pi - 1)\left\{2a\cos\frac{n\pi t}{\tau} + \frac{2n\pi}{\tau}\sin\frac{n\pi t}{\tau}\right\}\right].$$